On the distinguishability of downconverted modes with orbital angular momentum

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Received June 2, 2008; accepted July 7, 2008;
posted August 21, 2008 (Doc. ID 96842); published September 12, 2008

Assuming two quantum states of spontaneous parametric downconversion carrying orbital angular momentum, one may ask the question what is the minimum probability of error in identifying between two of these biphoton states by an arbitrary physical measurement over the biphoton state generated. While correctly chosen geometries may lead to perfect distinguishability of modes, it is worth noticing that experimental subtleties may lead to a poor mode distinguishability. We discuss the case where a restricted range instead of the needed range of wave vectors is collected by the experimental setup. These considerations may be useful for some applications, e.g., cryptography. © 2008 Optical Society of America

OCIS codes: 270.0270, 270.5568, 190.4410, 060.4785, 270.5585.

Aside from the fundamental importance of understanding entangled photon states carrying orbital angular momentum (OAM), these states have practical applications that include cryptography with a larger alphabet and potentialities for being used in quantum sensors for rotations and other devices. In physical cryptography, one may wish to diminish the attacker probability to identify between two states whereas giving the legitimate users A and B the means to a clear identification between them. Sometimes, a not well chosen measurement geometry—not always a trivial choice—may lead to a poor identification of states and may even destroy its usefulness. Helstrom's bound from information theory, coupled to basic elements of spontaneous parametric downconversion (SPDC), will be applied to states carrying OAM to study this question. Although this Letter is not aimed toward a specific application, a cryptography setup is proposed and sketched in Fig. 1 just to make a concrete case for the presented ideas.

A short-pulsed UV laser beam with a Gaussian profile is reconfigured by an OAM mask to a state with OAM l. The OAM mask is randomly reconfigurable to set one OAM value within M possible ones. The value l set at each laser pulse is the cryptographic key to be distributed between A and B. This reshaped UV beam excites a $\chi^{(2)}$ nonlinear crystal cut for Type II collinear SPDC. After UV deflection by a dichroic mirror (DM), the transmitted orthogonally polarized single and idler photons are separated and diverted to A and B detection systems. Photons received by A and B within a short Δt, the pulse duration time, at time t_j, are said to be "in coincidence." A randomly reconfigurable OAM mask is used by A before the detector. The detection system used by B may consist of an OAM static sorter including interferometers with Dove prisms (e.g., see [1]) and a single photon detector at each of the M outputs. The Dove sorter diverts the incoming photon to the detector port l_B. Over a public channel, A informs B the obtained value of l_A. B obtains $l=l_A+l_B$ that, in principle, is also known by A. The sequence of distinct l values obtained after the key distribution session constitute the shared keys within the "alphabet" of M possible values.

Even in this idealized setup there is the possibility of errors by A and B. Assume that B collects the light signals sent by A in a free propagation scheme and uses a too small collecting area at the Dove prism setup. To better understand this problem, consider the quantized electric field

$$\hat{\mathbf{E}}^{(+)}(\mathbf{r},t) = \sum_{\mathbf{k}} l_E(\mathbf{k})\mathbf{e}_{\mathbf{k}}\hat{a}_{\mathbf{k}} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_{\mathbf{k}}t)}, \qquad (1)$$

where $\mathbf{e}_{\mathbf{k}}$ is the unitary polarization vector and $l_E(\mathbf{k}) = -i\sqrt{\hbar\omega_{\mathbf{k}}/(2\epsilon V)}$. Also, consider the wave vector decomposition $U_l(\mathbf{r},t)$ of a spatial OAM mode, $U_l(\mathbf{r},t) = \Sigma_{\mathbf{k}} U_{\mathbf{k},l} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_{\mathbf{k}}t)}$, where $U_{\mathbf{k},l}$ are unitary matrices transforming between the two representations; they obey $\Sigma_l U_{\mathbf{k},l} U^*_{l,\mathbf{k}'} = \delta_{\mathbf{k},\mathbf{k}'}$. Writing $l_E(\mathbf{k})\mathbf{e}_{\mathbf{k}}\hat{a}_{\mathbf{k}} = \Sigma_{\mathbf{k}'} \delta_{\mathbf{k},\mathbf{k}'} l_E(\mathbf{k}')\mathbf{e}_{\mathbf{k}'}\hat{a}_{\mathbf{k}'}$ and replacing it in $\hat{\mathbf{E}}^{(+)}(\mathbf{r},t)$ one obtains

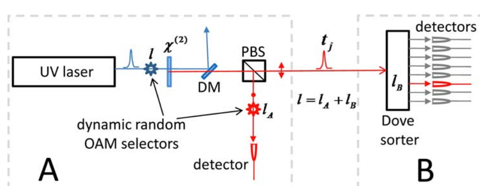

Fig. 1. (Color online) Cryptographic setup for quantum key distribution with OAM. One of the conjugate photons from SPDC generated with a pump mode with OAM l is sent from A to the user B. User A randomly detects a photon with OAM l_A; the conjugate photon is sent to B. A Dove sorter diverts the incoming photon to the detector port l_B. Over a public channel, A informs B the obtained value of l_A. B obtains $l=l_A+l_B$ that, in principle, is also known by A; l becomes a shared key in the OAM alphabet with M elements.

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$$\hat{\mathbf{E}}^{(+)}(\mathbf{r},t) = \sum_l \left[\sum_{\mathbf{k}'} U^*_{l,\mathbf{k}'} l_E(\mathbf{k}') \mathbf{e}_{\mathbf{k}'} \hat{a}_{\mathbf{k}'} \right] \times \left[\sum_{\mathbf{k}} U_{\mathbf{k},l} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_{\mathbf{k}}t)} \right] \equiv \sum_l \hat{\mathbf{c}}_l U_l(\mathbf{r},t). \quad (2)$$

This expression reminds one that to obtain a correct OAM mode a collection of wave vectors has to be allowed by the collecting system. Some experimental setups may restrict the collection angles to a very narrow span of wave vectors. One should note that the angular momentum index is not attached to a single wave vector $\mathbf{k}$ annihilation operator $\hat{a}_{\mathbf{k}}$ but to $\hat{\mathbf{c}}_l = \Sigma_{\mathbf{k}'} U^*_{l,\mathbf{k}'} l_E(\mathbf{k}') \mathbf{e}_{\mathbf{k}'} \hat{a}_{\mathbf{k}'}$, with all wave vectors needed to describe the involved mode. This Letter looks at cases where a poor distinguishability is set owing to a severe restriction on the collected wave vectors. The minimum probability of error in state identification is the quantitative measure adopted. A brief description of SPDC will be done before discussing the poor distinguishability of OAM.

Fundamentals of SPDC have been well studied for some decades; in a standard description the wave state is [2]

$$|\psi(t)\rangle = |0\rangle + \sum_{\sigma,\sigma'} \int d^3k' \int d^3k A_{\mathbf{k},\sigma;\mathbf{k}',\sigma'} l_E^{(*)}(\omega_{\mathbf{k}}) l_E^{(*)}(\omega_{\mathbf{k}}') \times T(\Delta\omega) \tilde{\psi}_{lp}(\Delta\mathbf{k}) \hat{a}^\dagger(\mathbf{k},\sigma) \hat{a}^\dagger(\mathbf{k}',\sigma')|0\rangle, \quad (3)$$

where $A_{\mathbf{k},\sigma;\mathbf{k}',\sigma'}$ is an amplitude containing the nonlinear susceptibility and unitary polarization vectors (σ polarized), $\tilde{\psi}_{lp}(\Delta\mathbf{k}) = \int_{V_I} d^3r \psi_{lp}(\mathbf{r}) \exp(-i\Delta\mathbf{k}\cdot\mathbf{r})$, and ψ_{lp} is the field amplitude in $\mathbf{E}(\rho,\phi,z;t) = \psi_{lp}(\mathbf{r}) e^{i(k_P z - \omega_P t)} \hat{\mathbf{e}}$. For a pump beam with OAM l [3], one has

$$\psi_{lp}(\rho,\phi,z) = \frac{A_{lp}}{\sqrt{1+(z/z_R)^2}} \left[\frac{\rho\sqrt{2}}{w(z)} \right]^l L_l^p \left[\frac{2\rho^2}{w(z)^2} \right] \times \exp\left[-\frac{\rho^2}{w(z)^2} \right] \exp\left[-i\left(\frac{k\rho^2 z}{2(z^2+z_R^2)} + l\arctan(y/x) \right) \right] \times \exp[i(2p+l+1)\arctan(z/z_R)]. \quad (4)$$

L_l^p is the associated Laguerre polynomial and $\rho^2 = x^2 + y^2$; $w(z)$ is the beam waist in a generic position z; z_R is the Rayleigh range, and $T(\Delta\omega) = \exp[i\Delta\omega(t - t_{\text{int}}/2)] \sin(\Delta\omega t_{\text{int}}/2)/(\Delta\omega/2)$ is the time window function defining the $\Delta\omega$ range given the interaction time t_{int}, $\Delta\omega = \omega_{\mathbf{k}} + \omega'_{\mathbf{k}} - \omega_P$, $\Delta\mathbf{k} = \mathbf{k} + \mathbf{k}' - \mathbf{k}_P$. The spectral function $\tilde{\psi}_{lp}(\Delta\mathbf{k})$ is central in defining SPDC and carries the specific pump mode that excites the nonlinear crystal. Using cylindrical coordinates (ρ,ϕ,z), where $dr^3 = \rho d\phi d\rho dz$, the integrals in (ρ,ϕ) in $\tilde{\psi}_{lp}(\Delta\mathbf{k})$ are performed giving [4] ($\rho_k = \sqrt{\Delta\mathbf{k}_x^2 + \Delta\mathbf{k}_y^2}$)

$$\tilde{\psi}_{lp}(\Delta\mathbf{k}) = \pi A_{lp} (i/2)^l (z_R/k_P)^{1+(l/2)} e^{-il(\pi/2)} \times \rho_k^l L_l^p[(z_R/k_P)\rho_k^2] e^{-(z_R/2k_P)\rho_k^2} e^{il\arctan(\Delta k_y/\Delta k_x)} \times \int_{z_0-l_c/2}^{z_0+l_c/2} e^{-i\Delta k_z z} e^{(-1+l+2p)\arctan(z_R/z)} \times e^{i(1+l+2p)\arctan(z/(2z_R))} dz. \quad (5)$$

With the usual condition $z_R \gg l_c$, one obtains ($\xi = (z_R/k_P)\rho_k^2$),

$$\tilde{\psi}_{lp}(\Delta\mathbf{k}) = \pi A_{lp}(-1) i (i/2)^l (l_c z_R/k_P) e^{i(p\pi - z_0\Delta k_z)} \times e^{il\arctan(\Delta k_y/\Delta k_x)} e^{-\xi/2} \xi^{l/2} L_l^p(\xi) \frac{\sin(l_c\Delta k_z/2)}{(l_c\Delta k_z/2)}. \quad (6)$$

$|\tilde{\psi}_{lp}(\Delta\mathbf{k})|^2$ is directly proportional to the crystal probability to generate signal and idler pairs with wave vectors $\mathbf{k}$ and $\mathbf{k}'$, given the incident pump field paraxial mode at $\mathbf{k}_P$. In this theory, the signal and idler fields are not limited to paraxial cases.

The problem we are interested in is the faithful discrimination between two SPDC states generated by two distinct pump modes set at (l,p) or (l',p'). Specification of the maximum of $|\tilde{\psi}_{lp}(\Delta\mathbf{k})|^2$ (phase matching), although being given for the variables ξ and Δk_z, imply correlated structures in the signal and idler wave vector space [4]; ξ can also be seen as a constraint over variables (k,k',ϕ,ϕ') imposed by a phase match in the nonlinear medium. The (apparent) simplicity of $|\tilde{\psi}_{lp}(\Delta\mathbf{k})|^2$ allows us to divide the phase-matching problem into longitudinal and transverse conditions, namely, $\sin(l_c\Delta k_z/2)/(l_c\Delta k_z/2) \sim 1$, value at $\Delta k_z = 0$, assumed within a width of $\Delta k_z \sim 2\pi/l_c$ (longitudinal condition) and $e^{-\xi_t}\xi_t^l L_l^p(\xi_t)^2$ = maximum (transverse condition); ξ_t is the value of ξ that maximizes $e^{-\xi}\xi^l L_l^p(\xi)^2$. The compact variable ξ connects the signal and idler transverse wave vectors as well as their dependence on the refractive indices in the medium. These dependences can be simple, as in Type I SPDC, or more complex, as in Type II SPDC (details can be seen in [4]).

Interference filters can provide a good degree of monochromaticity to signal and idler photons from SPDC. Their net effect on the SPDC spectrum will be described by the simplified filtered wave state (consider $p=0$), where the variables are taken around phase match points designated by the index pm:

$$|\psi(t)\rangle = \sin\theta'_{pm} {k'_{pm}}^2 \sin\theta_{pm} {k_{pm}}^2 A_{\mathbf{k}_{pm};\mathbf{k}'_{pm}} l_E^{(*)}(\omega_{\mathbf{k}_{pm}}) \times l_E^{(*)}(\omega_{\mathbf{k}'_{pm}}) \tilde{\psi}_{l0}(\Delta\mathbf{k}_{pm}) \hat{a}^\dagger_{l-l'}(\mathbf{k}_{pm}) \hat{a}^\dagger_{l'}(\mathbf{k}'_{pm})|0\rangle. \quad (7)$$

A too restricted collection of wave vectors among the allowed ones by Eq. (7) may cause that one photon in state $|1_1\rangle$ could not be distinguished from another one in state $|1_2\rangle$, where indices 1 and 2 designate two distinct OAM values for a downconverted photon, be-

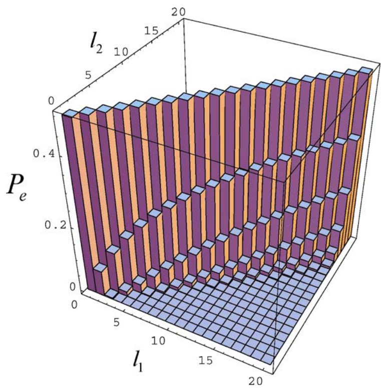

Fig. 2. (Color online) P_e in binary identification of Laguerre wave states with total biphoton angular momenta l_1 and l_2 under poor collecting conditions.

coming simply a detected photon $|1_{pm}\rangle$ around the phase-matching region. One may use analytical results from [4] modified to stress that this point $\tilde{\psi}_{lp}(\Delta\mathbf{k})$, written around the longitudinal phase match condition and assuming this poor distinguishability condition, is

$$\tilde{\psi}_{lp}(\Delta\mathbf{k}) = \left[-\frac{\pi}{2^l} i^{l+1} l_c A_{lp} \frac{z_R}{k_P} e^{il\ \arctan(\Delta k_y/\Delta k_x)} \right] \times e^{-\xi/2} \xi^{l/2} L_l^p(\xi) |1_{pm}\rangle |1'_{pm}\rangle \equiv g_{lp}(\Delta k_x, \Delta k_y, 0) e^{-\xi/2} \xi^{l/2} L_l^p(\xi) |1_{pm}\rangle |1'_{pm}\rangle. \quad (8)$$

Normalization of $\tilde{\psi}_{lp}(\Delta\mathbf{k})$ can be imposed by stating that a signal and idler pair will be found on the transverse plane. The phase term $e^{il\ \arctan(\Delta k_y/\Delta k_x)}$ can be simplified using degenerate SPDC that is expressed by $\rho=\rho'$, where $\rho=k \sin\theta$ and $\rho'=k' \sin\theta'$ for $k\simeq k'$ and $\theta\simeq\theta'$: $e^{il\ \arctan(\Delta k_y/\Delta k_x)} \rightarrow i^l$. Therefore, for degenerate SPDC, the normalized amplitude is

$$\tilde{\psi}_{lp}(\Delta\mathbf{k})_n = \tilde{\psi}_{lp}(\xi)_n = i(-1)^l \frac{e^{-\xi/2}\xi^{l/2}L_l^p(\xi)}{\sqrt{\int_0^\infty e^{-\xi}\xi^l L_l^p(\xi)^2 \mathrm{d}\xi}}. \quad (9)$$

One should also observe that other peculiarities may decrease efficiency in OAM detection and are not related to a poor collection system. It should be recalled that the OAM l is assigned to the biphoton as a whole [5,6]. A specific value l' of OAM may be assigned to one of the photons by means of OAM filters; in this case the conjugate photon takes the remaining OAM: $l-l'$. Depending on the symmetry of the medium, OAM does not show perfect conservation; this has been discussed in the literature (see details in [4]). This nonconservation possibility is not connected with the indistinguishability degree discussed in this Letter.

One should emphasize that usually cryptographic schemes do not repeat emissions of a given signal to avoid giving advantages to the eavesdropper. Consider that a single photon is contained at each laser pulse and that a specific wave vector is attached to the photon. As discussed, the signature, say l', of an OAM mode will not be attached to this single photon. In cases where the collected downconverted photons are wave vector identifiable leads to $|1_{l'}\rangle \rightarrow |1_{pm}\rangle$ and $|1_{l-l'}\rangle \rightarrow |1_{pm}\rangle$. In these cases, wave state overlap between two states indexed by l_j and l_k will only be specified by overlap of the probability amplitudes, namely,

$$\langle \psi_{l_j p_j} | \psi_{l_k p_k} \rangle \rightarrow \int_0^\infty \tilde{\psi}_{l_j p_j}(\xi)_n \tilde{\psi}_{l_k p_k}(\xi)_n \mathrm{d}\xi. \quad (10)$$

Helstrom's bound [7] applies to a binary decision between two pure arbitrary states $|\psi_0\rangle$ and $|\psi_1\rangle$. P_e is given by

$$P_e = \tfrac{1}{2}\left(1 - \sqrt{1 - 4p_0 p_1 |\langle \psi_0 | \psi_1 \rangle|^2}\right). \quad (11)$$

Figure 2 exemplifies the minimum probability of error in state determination between Laguerre modes with total angular momenta l_1 and l_2 with values in the interval $l=[1,20]$ when collecting systems are too restrictive for wave vectors. Equation (10) was used in P_e as given by Eq. (11) and taking the *a priori* probabilities $p_0=p_1=1/2$. The possibility for wave vector identification due to a poorly designed setup leads to a large fraction of uncontrollable errors by A and B and may easily render the system useless. Summarizing, quite general wave states describing SPDC with OAM were particularized to demonstrate how indistinguishability between OAM modes can be set.

This work was supported by the U. S. Army Research Office Multidisciplinary University Research Initiative grant W911NF-05-1-0197 on quantum imaging.

References

1. J. Leach, J. Courtial, K. Skeldon, S. M. Barnett, S. Franke-Arnold, and M. J. Padgett, Phys. Rev. Lett. **92**, 013601 (2004).
2. L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics* (Cambridge U. Press, 1995).
3. L. Allen, M. W. Beijersbergen, R. J. C. Spreeuw, and J. P. Woerdman, Phys. Rev. A **45**, 8185 (1992).
4. G. A. Barbosa, Phys. Rev. A **76**, 033821 (2007).
5. H. H. Arnaut and G. A. Barbosa, Phys. Rev. Lett. **85**, 286 (2000).
6. A. Mair, A. Vaziri, G. Weihs, and A. Zeilinger, Nature **412**, 313 (2001).
7. C. W. Helstrom, *Quantum Detection and Estimation Theory*, R. Bellman, ed. (Academic, 1976), p. 113, Eq. (2.34).