Heisenberg limited measurements with superconducting circuits

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We describe an assembly of N superconducting qubits contained in a single mode cavity. In the dispersive regime, the correlation between the cavity field and each qubit results in an effective interaction between qubits that can be used to dynamically generate maximally entangled states. With only collective manipulations, we show how to create maximally entangled quantum states and how to use these states to reach the Heisenberg limit in the determination of the qubit bias control parameter (gate charge for charge qubits, external magnetic flux for rf-SQUIDs).

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The description of the interaction between atoms and quantized modes of the electromagnetic field in a cavity is called cavity quantum electrodynamics (cQED). The first experimental studies used flying Rydberg atoms in an rf resonator [1]. With the advent of quantum computing several other implementations were developed to mimic the quantum properties of atoms. Among those, solid-state implementations are especially interesting since they offer several advantages over real atoms: these artificial atoms properties can be tailored and their number and location is fixed. We concentrate our discussion on superconducting circuits. Superconducting implementations of quantum computing attracted a lot of attention in recent years because they are inherently scalable and single qubit operations have been demonstrated with classical coherent control in a variety of qubits. Several theoretical studies treated the interaction of superconducting qubit(s) with a quantized electromagnetic field. The proposal of an on-chip cQED experiment using a Cooper-pair box as the artificial atom strongly coupled to a one-dimensional cavity [2] is especially interesting since it was followed buy several experiments. First, the strength of the coupling was shown to be indeed stronger than the different decays constants so that the vacuum Rabi splitting was observed [3]. Subsequently the ac-Stark shift was measured using a quantum non-demolition technique (QND) [4] and the decoherence time T_2 evaluated from Ramsey fringe experiments [5].

In a cavity with a high quality factor, the photons can serve as an information bus between several qubits and therefore create correlations between distant qubits. Beside its fundamental interest and applications to quantum information processing, entanglement offers the additional advantage of allowing improved sensitivity in a quantum-limited measurement. Compare to a measurement made with one sensor, an improvement of the order of $\sqrt{N}$ is obtained when N classical sensors are used to measure the same quantity. Now, if N quantum sensors are *coherently* coupled, the improvement is of the order of N. For instance, interferences between two different polarizations (modes) of three and four photon entangled states have been observed [6, 7] and spectroscopy performed on an assembly of three beryllium ions demonstrated a similar improvement in frequency estimation [8]. Today's solid-state sensors do not take advantage of this improvement made possible by quantum correlations. So using entanglement in a superconducting implementation to enable such Heisenberg limited measurements could revolutionize sensor technology with, for instance, electrometers and magnetometers.

In this work, we use techniques from other fields (atomic QED, ion traps, quantum optics) to describe interactions and manipulations needed in a superconducting circuit to perform a Heisenberg limited measurement. We propose to use the photon-qubit interaction to create an effective interaction between distant superconducting qubits. This interaction is used to generate maximally entangled states which in turn are used to beat the standard quantum limit when measuring the Ramsey frequency. We show how this results in an Heisenberg limited estimation of the qubit bias control parameter.

Most superconducting qubits can be described by the following single qubit Hamiltonian [9]:

$$H_Q = -\frac{B_z}{2}\,\sigma_z - \frac{B_x}{2}\,\sigma_x \tag{1}$$

with the bias B_z depending linearly of the control parameter λ: $B_z = b_z(1/2 - \lambda)$. This Hamiltonian is an approximation valid around a symmetry point, obtained for $\lambda = 1/2$, called the degeneracy point. Expressions for the parameters B_z, B_x, b_z and λ can be found in reference [9] in the case of a Cooper-pair box (CPB) or a rf-SQUID.

We now consider that the qubit is contained in cavity QED described by the Hamiltonian $H_c = \omega_c(a^\dagger a + 1/2)$. The quantized cavity mode adds an incremental contribution $\lambda_c(a^\dagger + a)$ to the bias. The single qubit Hamiltonian now reads:

$$H_{Q-C} = -\frac{B_z}{2}\sigma_z - \frac{B_x}{2}\sigma_x + \frac{b_z\lambda_c}{2}(a^\dagger + a)\sigma_z \qquad (2)$$

We assume the strong coupling limit $g \gg \kappa, \gamma$. We neglect the cavity decay κ and excited state decay γ for the moment. Experiments on a single CPB in a cavity [3] support this assumption ($g \approx 10\,\kappa$). We rewrite this Hamiltonian in the eigenbasis of (1) which is done by performing a rotation of θ around the y axis. The mixing angle θ is given by $\tan\theta = B_x/Bz$. The resulting individual contribution is then summed up N times to form the description of an assembly of N identical qubits in a single mode cavity:

$$\begin{aligned} H_N &= \Omega\, S_z + \omega_c(a^\dagger a + 1/2) \\ &-2g(a^\dagger + a)(S_z\cos\theta + S_x\sin\theta) \end{aligned} \qquad (3)$$

We used $2g = b_z\lambda_c$, $\Omega = \sqrt{B_x^2 + B_z^2}$ and $2\vec{S} = \sum_{i=1}^N \vec{\sigma_i}$. When $\theta = \pi/2$ and the rotating wave approximation is made, equation (3) take the form of the N-particle Jayne-Cummings Hamiltonian. When $\theta = 0$, equation (3) describes an harmonic oscillator with a conditional displacement, *e.g.* a displacement that depends on the total state of the system S_z. However, the limit $\theta \to 0$ is reached when the bias B_z is maximum and $B_x \to 0$. In this case, the two-level approximation (1), hence equation (3), is not justified. We assume that the qubit-cavity detuning $\Delta = \Omega - \omega_c$ is much larger than the qubit coupling g to the cavity. We write $S_x = (S_+ + S_-)/2$ and neglect the fast oscillating terms. Hamiltonian (3) can be approximately diagonalized with a polaronic transformation $U = \exp\big(\chi(aS_+ - a^\dagger S_-)\big)$:

$$\begin{aligned} \tilde{H}_N = U H_N U^\dagger &\approx \Omega\, S_z + \omega_c(a^\dagger a + 1/2) \\ &+\chi(S^2 - S_z^2 - S_z + 2a^\dagger a S_z) \end{aligned} \qquad (4)$$

We define $\chi = (g\sin\theta)^2/\Delta$. A more complete diagonalization would require to also take into account the displaced harmonic oscillator with $U_d = \exp(\frac{g\cos\theta}{\omega_c}(a - a^\dagger)S_z)$ and perform the transformation $U_d U H_N U^\dagger U_d^\dagger$. However, this complete transformation introduces terms proportional to g^2/ω_c which we neglect because we are interested in the regime where $g \ll \Delta \ll \omega_c$. We also neglected terms in χ^2 that would appear with the complete transformation. Hence, equation (4) is valid to the first order in χ. The two last terms of equation (4) are the shifts in the qubit and resonator frequencies, induced by the coupling of the qubits with the resonator. This Lamb and ac-Stark shift (respectively) were predicted in the case of a single CPB in a cavity [2].

The novelty of equation (4) compared to previously published results is the appearance of an effective interaction $H_{sz} = \chi S_z^2$ between N qubits mediated by a cavity photon. This interaction is a key element in non-optical implementations of Heisenberg limited estimations. It has been used extensively in ion traps experiments to generate maximally entangled states [10–12]. It was shown how to utilize these states to perform an optimal frequency measurement [13] and improve the estimation of rotation angles [14]. More recently, a method involving only collective manipulations has been used to perform precision spectroscopy on an assembly of six beryllium ions [15]. The first step consists to generate the maximally entangled states $|\psi_m\rangle$ (see for instance reference [16]) using the time evolution of H_{sz} over a time $t_{sz} = \pi/2\chi$:

$$\begin{aligned} |\psi_m\rangle &= e^{-i\frac{\pi}{2}S_z^2}|-N/2\rangle_x = \\ &\frac{1}{\sqrt{2}}\Big(|-N/2\rangle_x + i^{N+E}|+N/2\rangle_x\Big) \end{aligned} \qquad (5)$$

When the number of qubits N is odd, another rotation $e^{i\frac{\pi}{2}S_z}$ is needed in addition to the $e^{i\frac{\pi}{2}S_z^2}$ [16]. We take the parity into account in equation (5) by setting the quantity E to 2 (1) when N is odd (even resp.). Initially all the qubits are in their ground state, $|J = N/2, M = -N/2\rangle_z = |-N/2\rangle_z = |\downarrow\rangle_1|\downarrow\rangle_2\cdots|\downarrow\rangle_N$ (we do not consider the field part of the wave function at this point). As the one-axis twisting term in equation (4) is defined with the z axis, we need to perform a $\pi/2$ rotation (around x) so that the twisting can take place. Therefore, we define the operator $U_N = e^{i\frac{\pi}{2}S_x}e^{-iH_{sz}t_{sz}}e^{-i\frac{\pi}{2}S_x}$ for notation convenience. The average of spin vector calculated with the wave function (5) is zero, $\langle\vec{S}\rangle = 0$, so that the natural choice of S_z ($\vec{S}$) as the observable to be measured can not be made. Bollinger *et al.* showed that the parity operator $\prod_i \sigma_{z_i}$ was an adequate observable that could be measured with the state (5). However the measurement of this operator for a large number N of qubits is difficult since it requires to distinguish odd and even numbers of particles in state $|\downarrow\rangle$. A method involving only collective manipulations as been proposed in [8] to circumvent this problem. First the maximally entangled state is constructed. In our case the sequence U_N is applied to the initial state $|-N/2\rangle_z$. Afterward, the system acquires a phase ϕ during a free evolution period T, which here is given by $\Omega \times T$ (for the purpose of the phase determination, we neglect the Lamb shift). Finally, another application of the sequence U_N transfers the phase information, $N\phi/2$, into an amplitude information of either state $|+N/2\rangle_z$ or $|-N/2\rangle_z$:

$$\begin{aligned} |\psi\rangle &= U_N e^{-i\phi} U_N|-N/2\rangle_z = \\ &-i\sin\Big(\frac{N}{2}\,\phi\Big)|-N/2\rangle_z + \\ &i^{N+E}\cos\Big(\frac{N}{2}\,\phi\Big)|+N/2\rangle_z \end{aligned} \qquad (6)$$

The measurement will collapse the wave function $|\psi\rangle$ on the state $|+N/2\rangle_z$ with probability $P_\uparrow = \frac{1}{2}(1+\cos(N\phi))$.

We propose to use cavity spectroscopy to infer the state of the qubits. Assuming now there is finite but small cavity decay rate κ, a signal at frequency ω_c sent in the cavity will experience a phase shift when it is transmitted. Solving the Heisenberg equation for the field creation operator, this phase shift ϑ is given by $\tan\vartheta = \pm(2\chi N)/\kappa$. The probability $P_\uparrow$ is extracted from the time dependence of ϑ. A measurement scheme as been proposed based on this principle to perform a quantum non-demolition measurement of the state of a single Cooper-pair box contained in a cavity QED [2]. The difference is that because the coupling to the cavity is $\sqrt{N}$ stronger, the phase shift ϑ is more important than the single qubit case.

The main motivation to use N-particle maximally entangled state to perform a spectrocospy measurement is to be able to relate the N fold frequency increase to the phase uncertainty. Uncertainty on a parameter ζ can be estimated from the error propagation formula $\delta\zeta = \Delta\hat{A}/|\partial\langle\hat{A}\rangle/\partial\zeta|$ by measuring the operator $\hat{A}$. We introduce the projection operator $\hat{A} = |+N/2\rangle\langle+N/2|$ so that the quantity we propose to measure $P_\uparrow$ is the average of $\hat{A}$ over the state $|\psi\rangle$ of equation (6). The variance $\Delta\hat{A}^2$ is then simply given by $P_\uparrow(1-P_\uparrow)$ (second moment of the Bernouilli distribution). Hence, the measurement of $P_\uparrow$ leads to an estimation of the phase uncertainty $\delta\phi$ equal to $1/N$. Neglecting the frequency shift, the phase acquired during the free evolution is $\Omega \times T$, and therefore the frequency uncertainty is given by $\delta\Omega = 1/(NT)$. In superconducting circuits, the parameter λ controls the level spacing Ω and therefore the uncertainties of both quantities can be related through the following relation:

$$\delta\Omega = 2b_z \cos\theta\,\delta\lambda \tag{7}$$

Hence, a Heisenberg limited measurement of the frequency Ω performed with the sequence (6) results in an improvement of the estimation of the parameter $\delta\lambda = 1/(NT \times 2b_z \cos\theta)$. This method can be used to improve the estimation of the gate charge n_g (or the external bias flux ϕ_x) in a system composed of N Cooper-pair boxes (rf-SQUIDs resp.) coherently coupled. To help understanding our scheme in particular and Heisenberg limited measurements in general, we wish to emphazise the difference between the quantum observable measured and the parameter that the method allows to determine with a better precision. The quantum variable defines the types of superconducting sensors or qubit. To simplify the discussion, let's say there are mainly two types of superconducting devices, electric charge and magnetic flux sensors. The quantum variable measured is then either the charge $\hat{n}$ or the magnetic flux $\hat{\phi}$. Among several characteristics, the performance of the device is set partly by the quantum fluctuations of this quantity also refered to as shot-noise. Now, the Hamiltonian of the system can be tuned or controled with a classical (continuous) variable which is the gate charge n_g in a charge qubit or the external magnetic flux ϕ_x in a flux qubit. A Heisenberg limited measurement consists in evaluating the uncertainty of the parameter (δn_g or $\delta\phi_x$) for a given value of this parameter with a measurement of the quantum variable ($\hat{n}$ or $\hat{\phi}$). Sometimes in the literature, the error on the parameter obtained with N qubits *uncoherently* coupled is called shot-noise limit (known also as classical quantum limit, CQL) as it scales as $1/\sqrt{N}$. This designation is unfortunate as it can introduce a confusion between the shot-noise of a *single* device refering to the fluctuations of either the electric charge or the magnetic flux, and the fluctuations of this quantity as whole when it is actually spread over many devices.

Our scheme is useful *away* from the degeneracy point as at this point $\cos\theta = 0$. However, one should bear in mind that the coupling χ decreases as the operating point is moved away from the degeneracy point, so a tradeoff should be made to operate away, while no too far from this point. The limited validity range around the degeneracy point is not a limitation in a system composed of N Cooper-pair boxes since the different coherence times decrease with the distance from this point [17] and therefore the operation far away from it, is not adequat to observe coherent effects.

In this work, we described a collection of N superconducting qubits contained in a single mode cavity. Beside the usual shifts in the qubit and resonator frequencies, we find that the effective Hamiltonian contains a term χS_z^2 describing the interaction between all the qubits. This interaction can be used to dynamically prepare maximally entangled states. We adapt a method used in ion traps to demonstrate the use of these states to reach the Heisenberg limit in the Ramsey frequency determination. Finally, we show that the parameter that controls the energy spacing can be estimated with an uncertainty that scales inversely with the number N of qubits.

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