The Institute of OPTICS

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Effects of Turbulence on the Transverse Position-Momentum Entanglement of Biphotons

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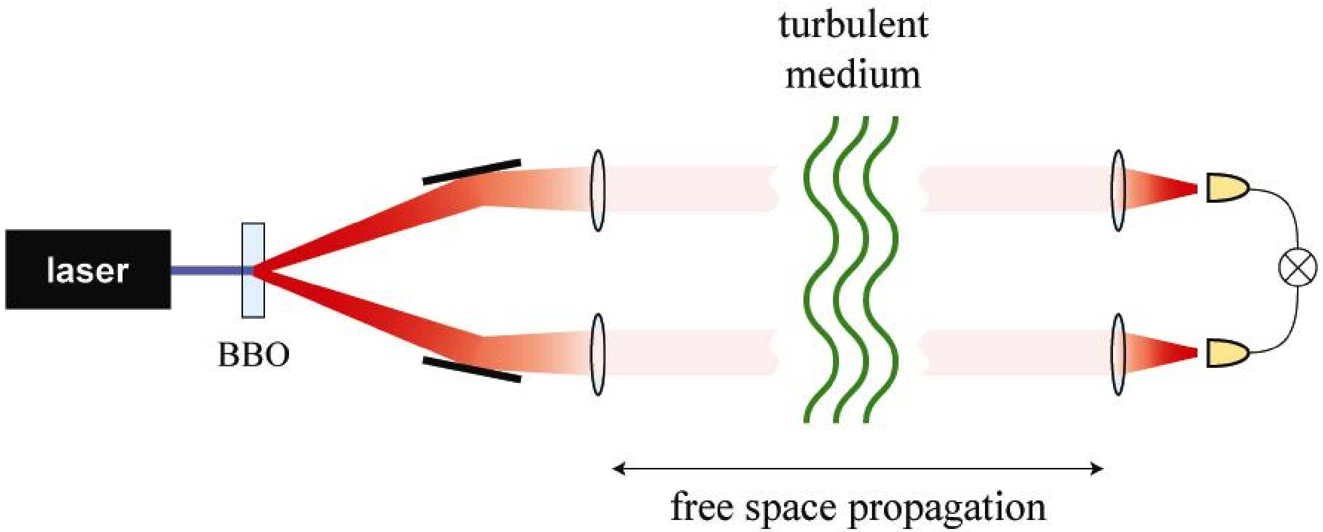
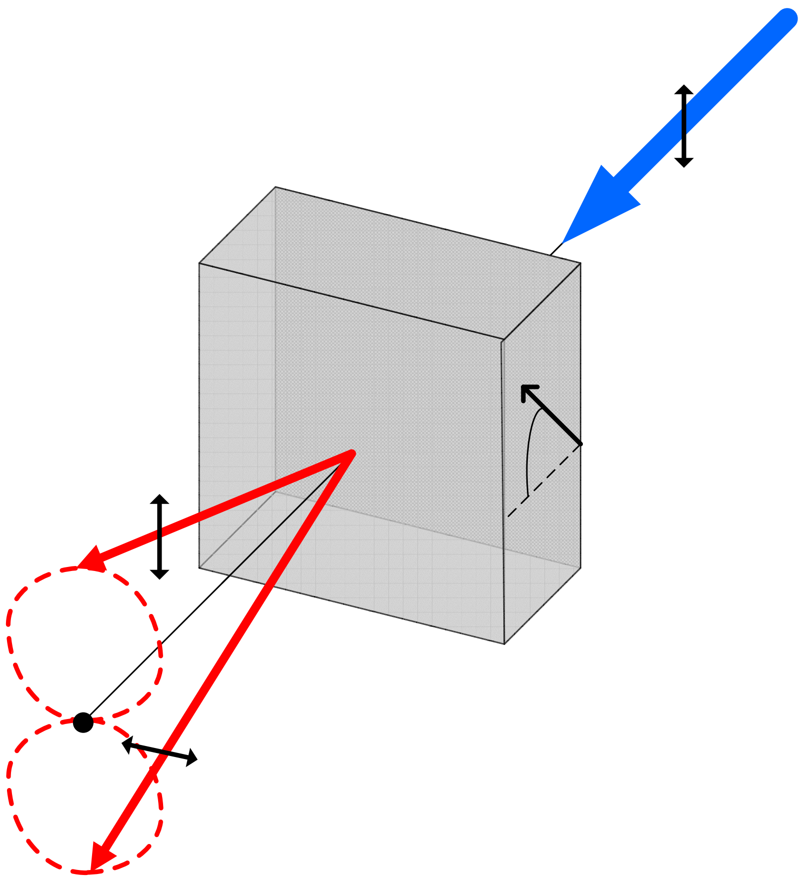
Motivations

1. Entanglement provides secure information transmission
2. High dimensional Hilbert space (*qubits – only 2D*)
   - time-energy
   - transverse spatial coordinates of biphotons
     - ❑ orbital angular momentum (OAM)
     - ☑ transverse linear position / momentum (quantum image)
3. Long distance communication in free space (turbulence effect)

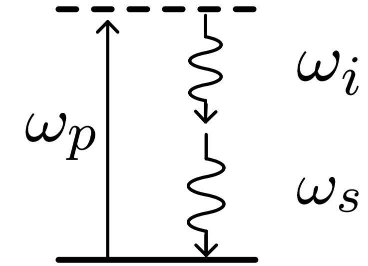


Generation of Entangled Photons

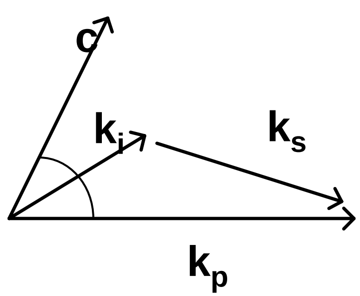
Type-II Spontaneous Parametric Down-Conversion (SPDC)



$$\omega_p = \omega_s + \omega_i$$



$$\mathbf{k}_{\mathbf{p}}^{(e)} = \mathbf{k}_{\mathbf{s}}^{(o)} + \mathbf{k}_{\mathbf{i}}^{(e)}$$



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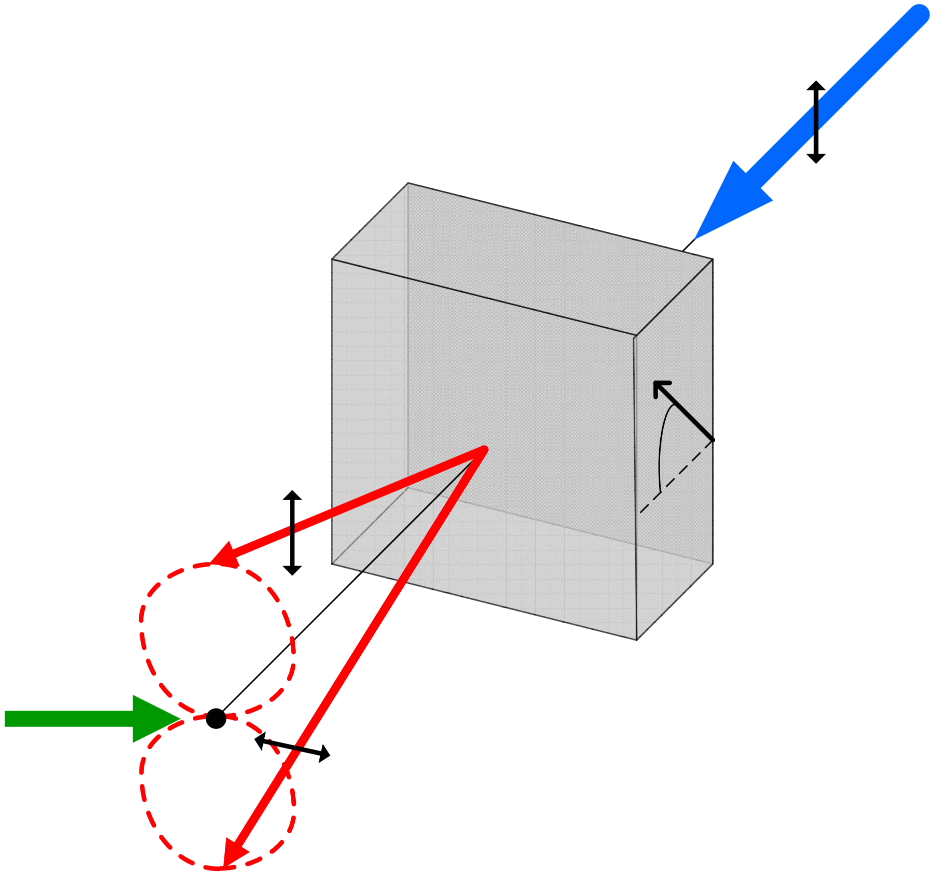
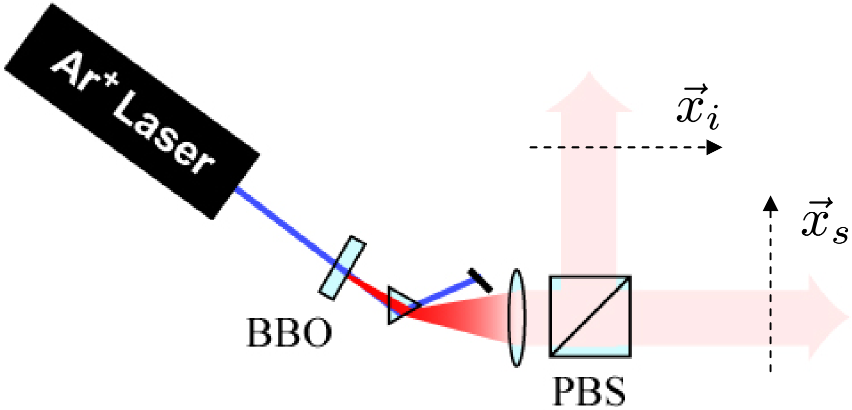


We will restrict ourselves to degenerate and nearly collinear SPDC taking,

$$2\omega_s = 2\omega_i = \omega_p$$

$$2k_s \simeq 2k_i \simeq k_p$$

Theory of SPDC (Gaussian Approx. 1)



The biphoton state is given by

$$|\Psi\rangle = \int d\vec{p}_s \; d\vec{p}_i \;\; \Phi(\vec{p}_s, \vec{p}_i) \; a_s^\dagger(\vec{p}_s) a_i^\dagger(\vec{p}_i)|0,0\rangle$$

with the wave function in momentum space

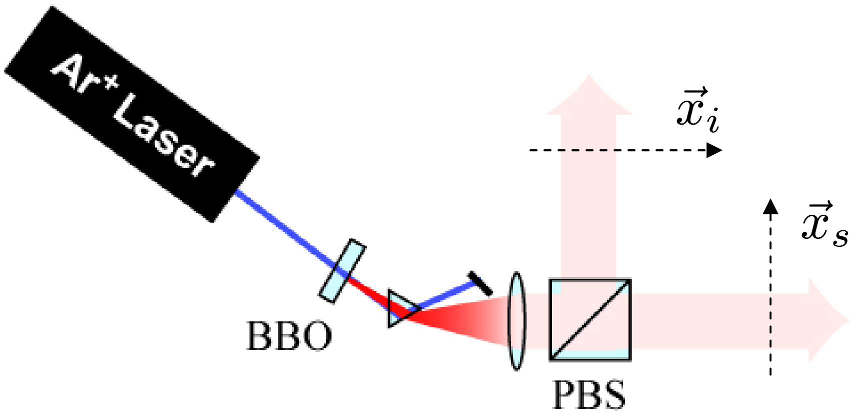
$$\Phi(\vec{p}_s, \vec{p}_i) = N E_p(\vec{p}_s + \vec{p}_i) \; \mathrm{sinc}\left(\frac{\Delta_k L}{2}\right) \exp\left(i \frac{s_k L}{2}\right)$$

where N is a normalization factor,

$\Delta_k = k_p(\vec{p}_s + \vec{p}_i) - k_s(\vec{p}_s) - k_i(\vec{p}_i)$ and $s_k = k_p(\vec{p}_s + \vec{p}_i) + k_s(\vec{p}_s) + k_i(\vec{p}_i)$

E_p is the transverse profile of the pump.

Theory of SPDC (Gaussian Approx. 1)



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$$\rightarrow N \exp\left[-\frac{1}{2B}(\vec{p}_s + \vec{p}_i)^2\right] \exp\left[-\frac{1}{2A}(\vec{p}_s - \vec{p}_i)^2\right]$$

Law & Eberly, PRL **92**, 127903 (2004)

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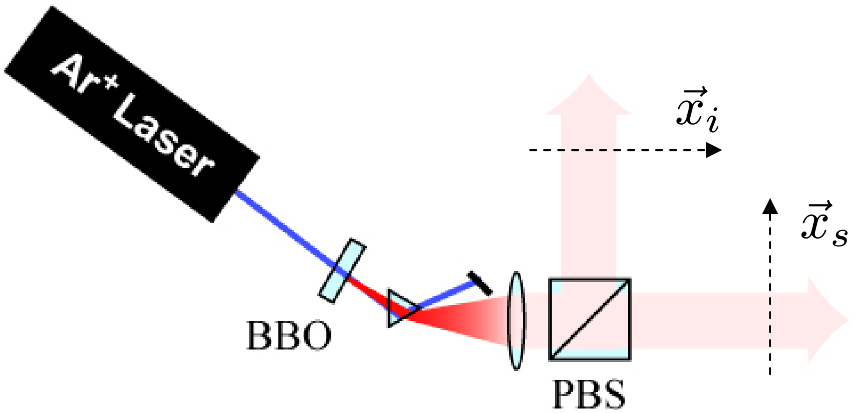

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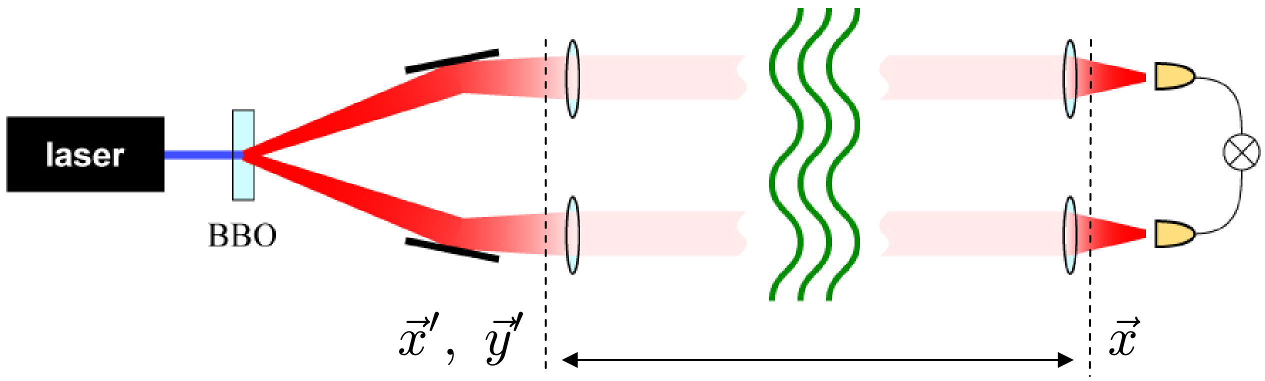
(position space)

$$\Psi(\vec{x}_s, \vec{x}_i) \rightarrow N \exp\left[-\frac{B}{2}(\vec{x}_s - \vec{x}_i)^2\right] \exp\left[-\frac{A}{2}(\vec{x}_s + \vec{x}_i)^2\right]$$

Law & Eberly, PRL **92**, 127903 (2004)

Theory of Turbulence



The propagated field in a turbulent medium is given by [Ref: Milonni, AJP **67**, 476(1999)]

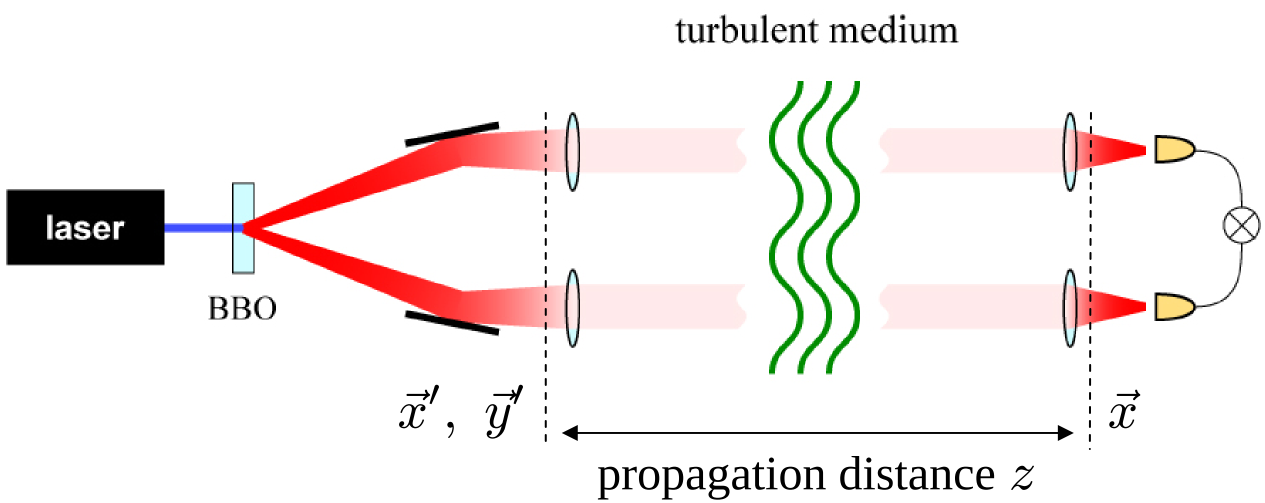
$$\hat{E}^{(+)}(\vec{x}, z) = e^{ikz} \int d\vec{x}' \; h(\vec{x}, \vec{x}', z) e^{i\phi(\vec{x}')} \hat{E}^{(+)}(\vec{x}', 0)$$

where $$h(\vec{x}, \vec{x}', z) = \frac{k}{2\pi i z} \exp\left[\frac{ik}{2z}(\vec{x} - \vec{x}')^2\right]$$ (paraxial limit)

The field at z = 0 is given by the Fourier transform of the annihilation operator

$$\hat{E}^{(+)}(\vec{x}, 0) \sim \int d\vec{q} \; \hat{a}(\vec{q}) \; e^{i\vec{q}\cdot\vec{x}}$$

Theory of Turbulence



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<u>Note</u>:

1. Turbulent medium is reflected from the statistical character of $\phi(\vec{x}')$.

2. The medium is replaced by a single "phase screen" accounting for all the phase fluctuation incurred in the propagation to z, i.e., $\phi(\vec{x}') = k \int_0^z n(\vec{x}', z')dz'$

3. Fluctuating phase: $\overline{e^{i[\phi(\vec{x}') - \phi(\vec{y}')]}} = e^{-(1/2)D_s(|\vec{x}' - \vec{y}'|)}$
   Phase structure function $D_s(|\vec{x}' - \vec{y}'|) = \alpha|\vec{x}' - \vec{y}'|^{5/3}$ (Kolmogorov)

SPDC + Turbulence

Four-point correlation function

$$G(\vec{x}_s,\vec{y}_s;\vec{x}_i,\vec{y}_i) \equiv \langle\Psi|\overline{\hat{E}^{(-)}(\vec{y}_i,z_i)\hat{E}^{(-)}(\vec{y}_s,z_s)\hat{E}^{(+)}(\vec{x}_s,z_s)\hat{E}^{(+)}(\vec{x}_i,z_i)}|\Psi\rangle$$
$$= \iiiint d\vec{x}_s'\, d\vec{y}_s'\, d\vec{x}_i'\, d\vec{y}_i' \;\; T(\cdots)\times G_0(\vec{x}_s',\vec{y}_s';\vec{x}_i',\vec{y}_i')$$

Free space transfer function

$$T(\cdots) \equiv h(\vec{x}_s-\vec{x}_s',z_s)h^*(\vec{y}_s-\vec{y}_s',z_s)h(\vec{x}_i-\vec{x}_i',z_i)h^*(\vec{y}_i-\vec{y}_i',z_i)$$

All entanglement information is contained in

$$G_0(\vec{x}_s,\vec{y}_s;\vec{x}_i,\vec{y}_i) = e^{-\frac{1}{2}\mathcal{D}_s(|\vec{x}_s-\vec{y}_s|)}e^{-\frac{1}{2}\mathcal{D}_i(|\vec{x}_i-\vec{y}_i|)}\Psi(\vec{x}_s,\vec{x}_i)\Psi^*(\vec{y}_s,\vec{y}_i)$$

Remark:

The turbulent medium induces local decoherence:

a) pure state $\rightarrow$ mixed state

b) non-unitary $\Rightarrow$ disentanglement?

Quantification of Entanglement

Biphoton density matrix

$$G_0(\vec{x}_s, \vec{y}_s; \vec{x}_i, \vec{y}_i) = e^{-\frac{1}{2}\mathcal{D}_s(|\vec{x}_s - \vec{y}_s|)} e^{-\frac{1}{2}\mathcal{D}_i(|\vec{x}_i - \vec{y}_i|)} \Psi(\vec{x}_s, \vec{x}_i) \Psi^*(\vec{y}_s, \vec{y}_i)$$

with $$\Psi(\vec{x}_s, \vec{x}_i) = N \exp\left[-\frac{B}{2}(\vec{x}_s - \vec{x}_i)^2\right] \exp\left[-\frac{A}{2}(\vec{x}_s + \vec{x}_i)^2\right]$$

$$D_s(r) = D_i(r) = \alpha r^{5/3}$$

For CV entanglement

The second moments of the variables provide useful information about the degree of entanglement. #

Measures of entanglement

1. EPR uncertainty
2. Entanglement of formation

Hyllus & Eisert, New J. Phys. **8**, 51 (2006)

Gaussian Approximation 2

Biphoton density matrix

$$G_0(\vec{x}_s,\vec{y}_s;\vec{x}_i,\vec{y}_i) = e^{-\frac{1}{2}\mathcal{D}_s(|\vec{x}_s-\vec{y}_s|)} e^{-\frac{1}{2}\mathcal{D}_i(|\vec{x}_i-\vec{y}_i|)} \Psi(\vec{x}_s,\vec{x}_i)\Psi^*(\vec{y}_s,\vec{y}_i)$$

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$$D_s(r) = D_i(r) = \alpha r^p, \qquad p = \frac{5}{3}$$

Covariance matrix

$$\Delta^2 x_s = \Delta^2 x_i = \frac{1}{4}\left(A^{-1}+B^{-1}\right)$$

$$\Delta^2 p_s = \Delta^2 p_i = A + B + \alpha p(p-1)\epsilon^{p-2} - \frac{(\alpha p)^2}{2}\epsilon^{2(p-1)}, \quad \epsilon \to 0$$

$$\Delta x_s \Delta x_i = \frac{1}{4}\left(A^{-1}-B^{-1}\right)$$

$$\Delta p_s \Delta p_i = A - B$$

$$\Delta x_s \Delta p_s = \Delta x_i \Delta p_i = \Delta x_s \Delta p_i = \Delta x_i \Delta p_s = 0$$

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Singularity in $\Delta^2 p$

Gaussian Approximation 2

Biphoton density matrix

$$G_0(\vec{x}_s, \vec{y}_s; \vec{x}_i, \vec{y}_i) = e^{-\frac{1}{2}\mathcal{D}_s(|\vec{x}_s - \vec{y}_s|)} e^{-\frac{1}{2}\mathcal{D}_i(|\vec{x}_i - \vec{y}_i|)} \Psi(\vec{x}_s, \vec{x}_i) \Psi^*(\vec{y}_s, \vec{y}_i)$$

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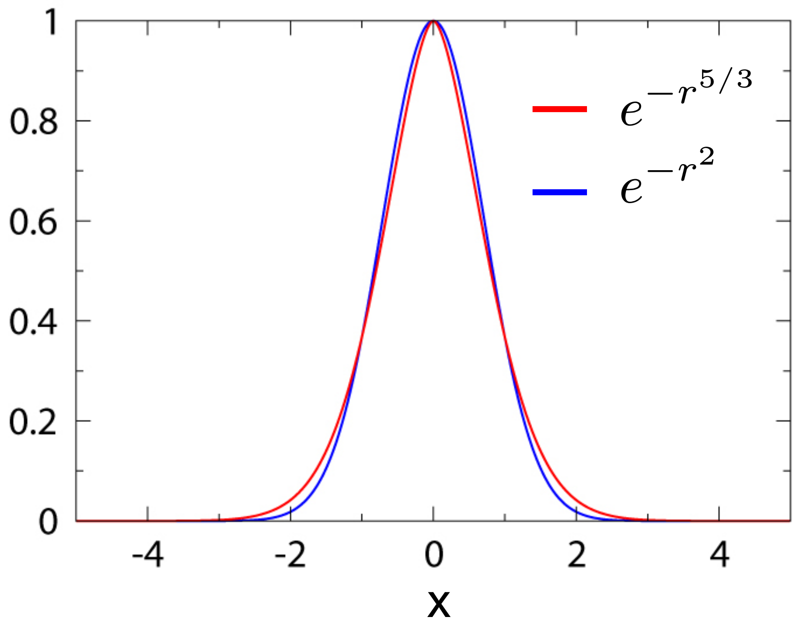
$$D_s(r) = \alpha\, r^{5/3} \xrightarrow{?} \alpha\, r^{6/3}$$



Expect their impacts to entanglement degradation are very similar.

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$$D_s(r) = \alpha\, r^{5/3} \xrightarrow{\text{OK}} \alpha\, r^{6/3}$$

Covariance matrix

$$\Delta^2 x_s = \Delta^2 x_i = \frac{1}{4}\left(A^{-1} + B^{-1}\right)$$

$$\Delta^2 p_s = \Delta^2 p_i = A + B + 2\alpha$$

$$\Delta x_s \Delta x_i = \frac{1}{4}\left(A^{-1} - B^{-1}\right)$$

$$\Delta p_s \Delta p_i = A - B$$

$$\Delta x_s \Delta p_s = \Delta x_i \Delta p_i = \Delta x_s \Delta p_i = \Delta x_i \Delta p_s = 0$$

No singularity

Control Parameters

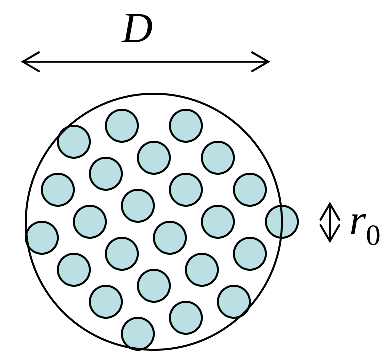
Phase structure function (Kolmogorov + Gaussian approx.)

$$D_s(r) = 3.44\left(\frac{r}{r_0}\right)^{6/3}$$

r_0 – the length scale of turbulent structure

Effective aperture diameter:

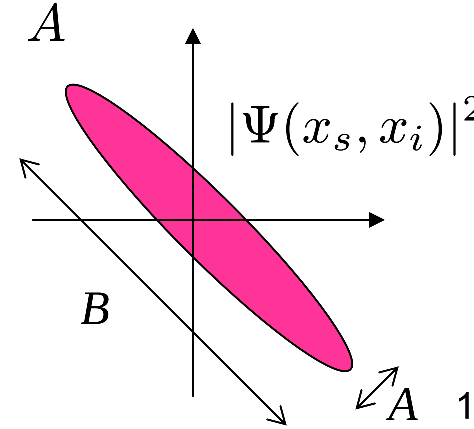
$$D = 2\Delta x_s = 2\Delta x_i = \sqrt{A^{-1} + B^{-1}}$$



Control parameters – (i) degree of turbulence: $\dfrac{D}{r_0}$

(ii) degree of initial entanglement: $\eta = \dfrac{B}{A}$

$$\Psi(\vec{x}_s, \vec{x}_i) = N \exp\left[-\frac{B}{2}(\vec{x}_s - \vec{x}_i)^2\right] \exp\left[-\frac{A}{2}(\vec{x}_s + \vec{x}_i)^2\right]$$



Effect of Turbulence to Entanglement

EPR uncertainty

$$\Delta = \sqrt{\Delta^2(x_s - x_i) + \Delta^2(p_s + p_i)}$$

In general: $\Delta < 1$ entangled

$\Delta \geq 1$ entangled / disentangled

Giedke *et al*., PRL **91**, 107901 (2003)

Effect of Turbulence to Entanglement

EPR uncertainty

$$\Delta = \sqrt{\Delta^2(x_s - x_i) + \Delta^2(p_s + p_i)}$$

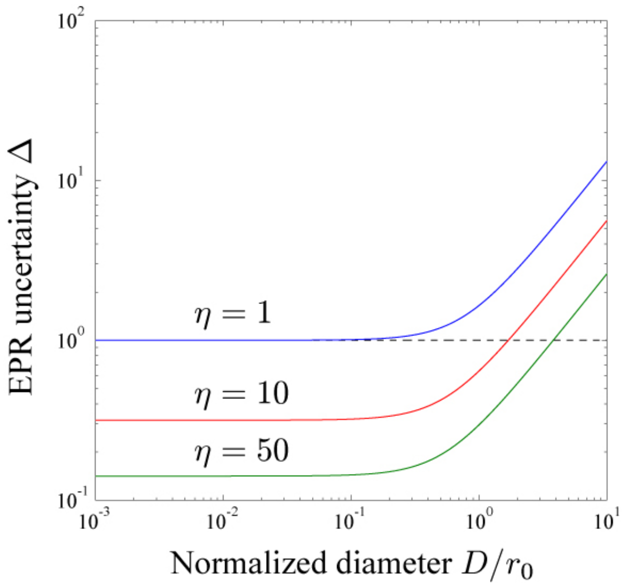
Gaussian state:

$\Delta < 1$ entangled

$\Delta \geq 1$ disentangled

We find #

$$\Delta = \sqrt{\frac{(1+\eta^{-1}) + 3.44(D/r_0)^2}{1+\eta}}$$



Note:

1. Entanglement of the two photons is totally destroyed if

$$3.44\left(\frac{D}{r_0}\right)^2 \geq \eta - \eta^{-1}$$

2. $$2\log\Delta = \log\left[1 + \frac{3.44}{1+\eta^{-1}}\left(\frac{D}{r_0}\right)^2\right] + \log\left(\frac{1+\eta^{-1}}{1+\eta}\right)$$

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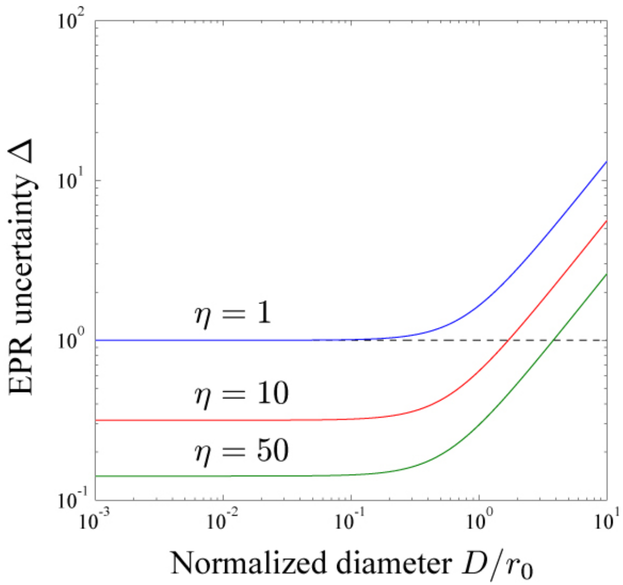
$$\Delta = \sqrt{\frac{(1+\eta^{-1}) + 3.44(D/r_0)^2}{1+\eta}}$$

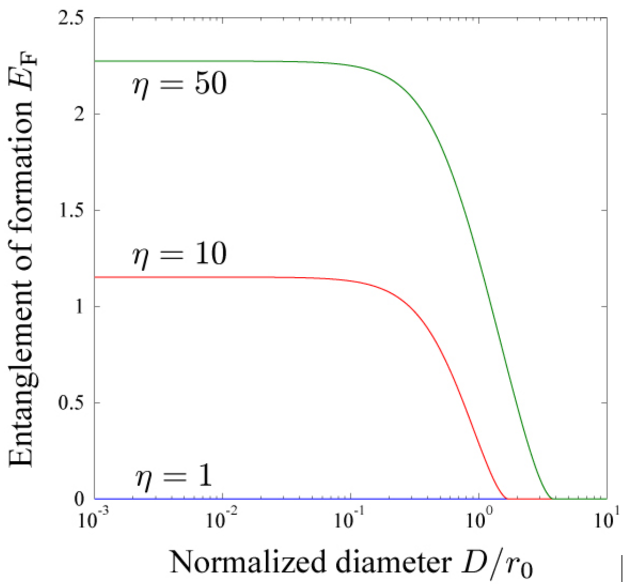
<u>Entanglement of formation for Gaussian states</u>

– how much entanglement is needed to construct the state

$$E_F = c_+ \log c_+ - c_- \log c_-$$

where $$c_\pm = \frac{1}{4}\left(\Delta^{-1/2} \pm \Delta^{1/2}\right)^2$$



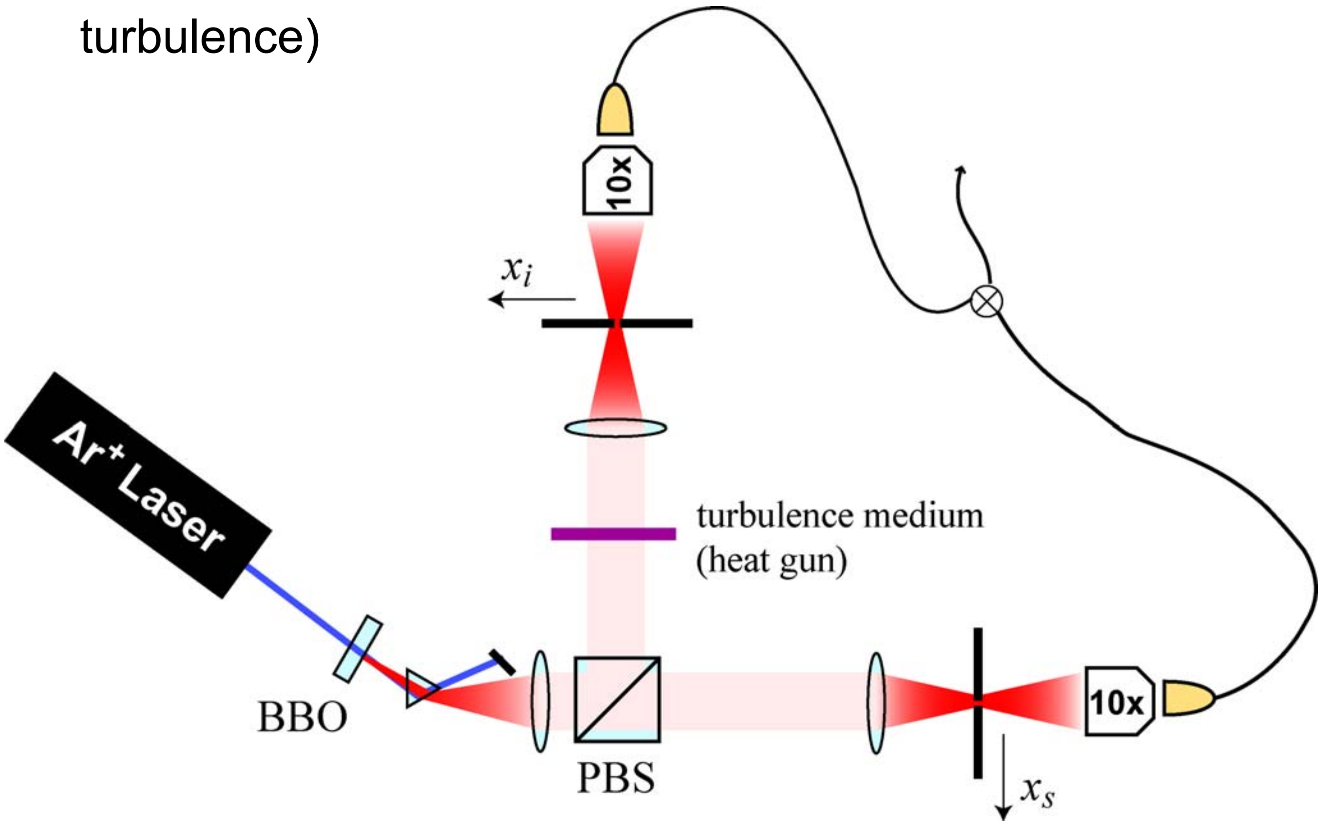


Giedke *et al.*, PRL **91**, 107901 (2003)

Experiment & Preliminary Results

Turbulence medium:

- ❑ heat gun
  (easy to implement)
- ❑ Kolmogorov phase screen
  (quantitative degree of turbulence)



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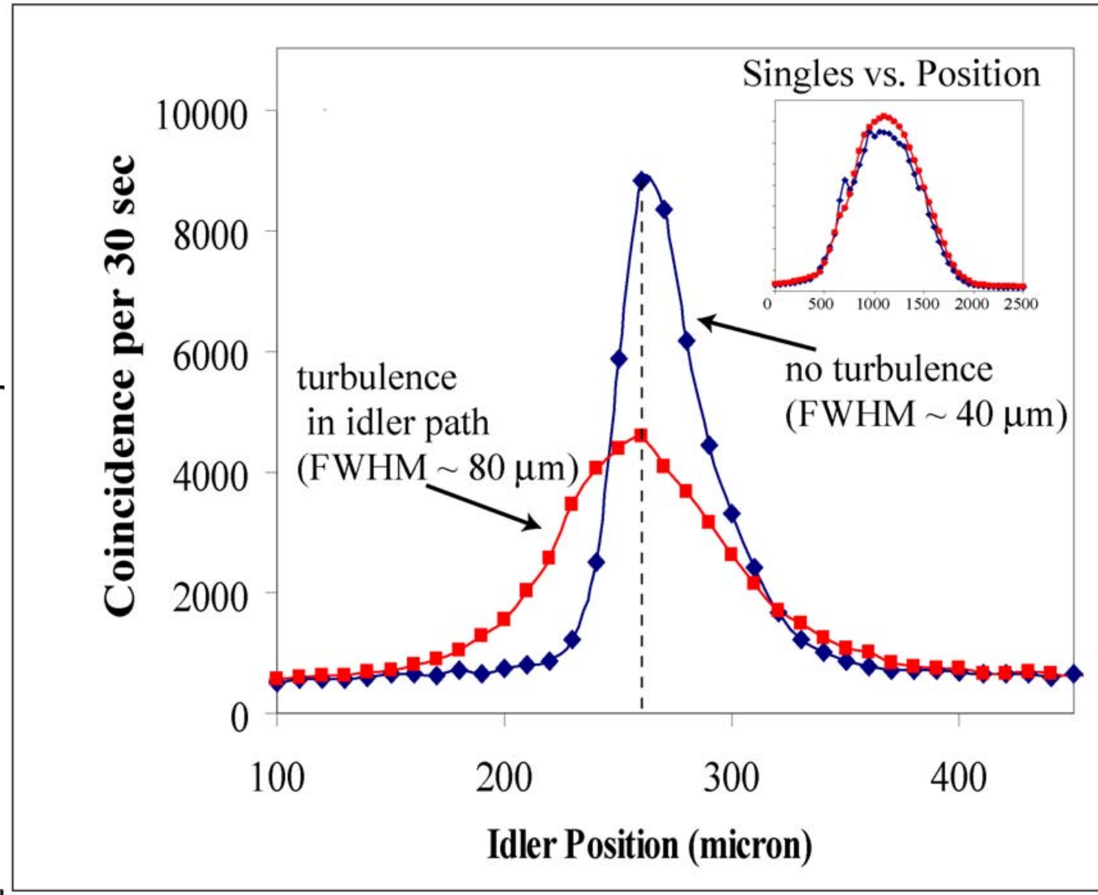

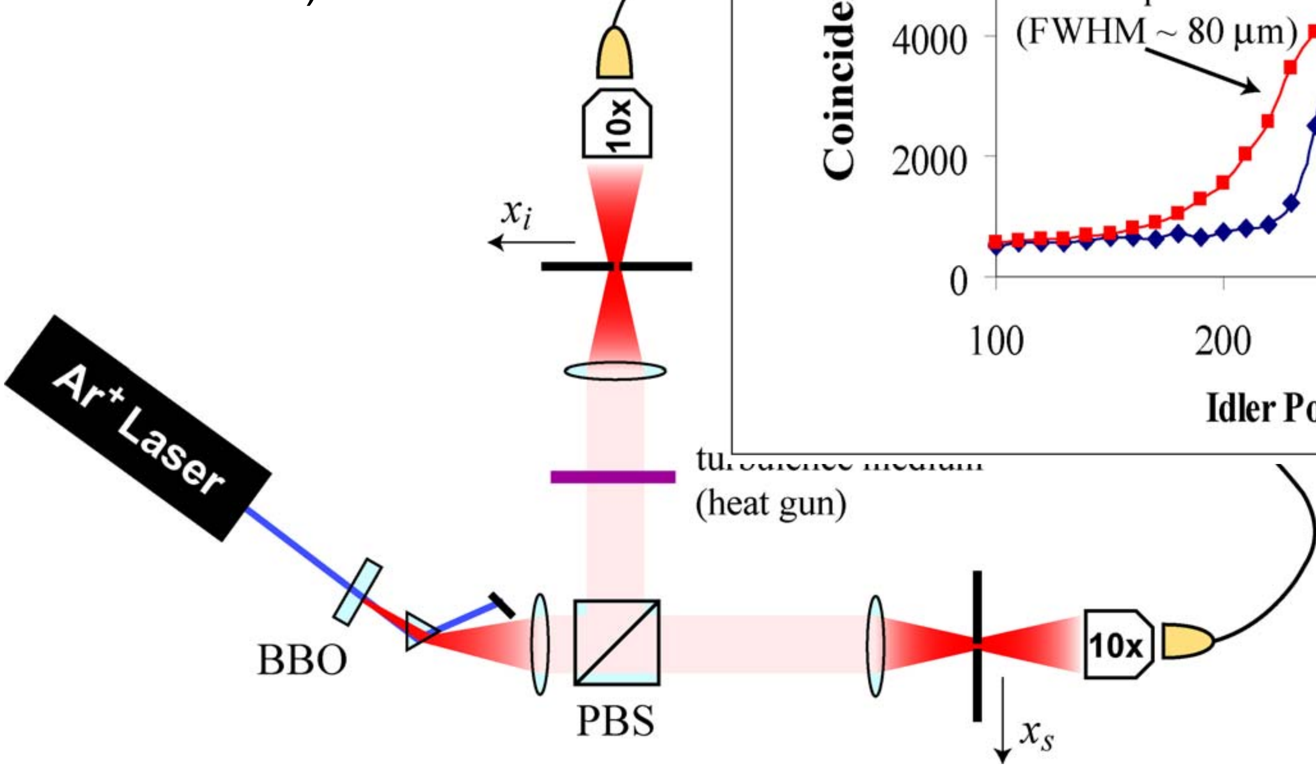

Summary / Outlook

Summary

- Constructed the theory of SPDC spatial coordinates with turbulence effect
- Approximated the system by a Gaussian model:
  - *initial wave function* (position & momentum coordinates)
  - *turbulence power law* (phase structure function: 5/3 → 6/3)
- Shown effect of turbulence of the entanglement. Entanglement is strongly affected at $D/r_0 > 0.5$.
- Obtained preliminary experimental results

Next steps

- Quantitative measurement of entanglement vs. turbulence effect
- Use adaptive optics to minimize disentanglement

Thank you

Q & A

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- US Army Research Office under a MURI grant
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