# Optimal Measurement of Multimode Squeezed Light via Eigenmode Analysis

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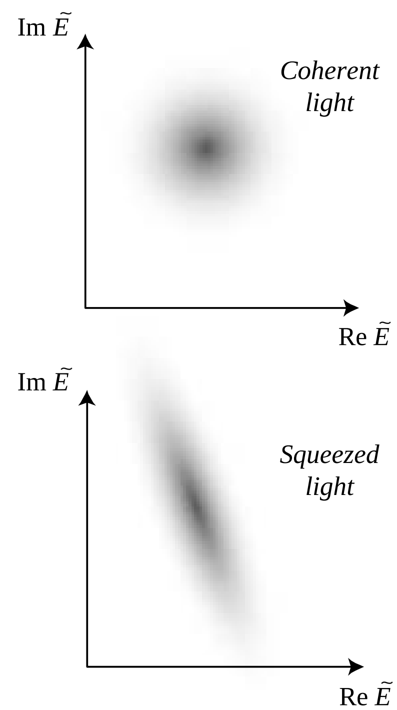
# The quantum uncertainty of light

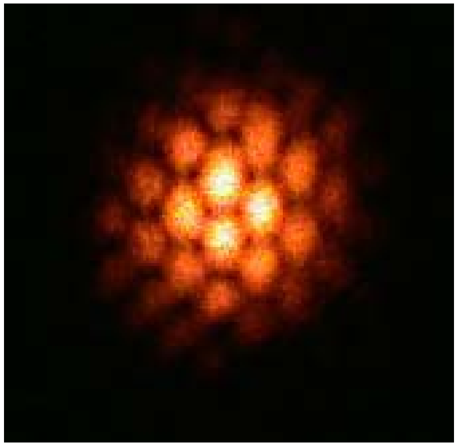
- The quantum nature of light prevents the amplitude and phase of an optical field from simultaneously having precise values.
- This quantum uncertainty is manifest as photocurrent (shot) noise in optical detection. It represents a fundamental limit of precision in optical measurement.
- Coherent two-photon emission produces light whose complex amplitude has less quantum uncertainty in one quadrature than the other. This is *squeezed light*[1].




*Characteristic probability distributions for measurement of coherent and squeezed light.*

# Squeezed light in the lab

- Squeezed light holds promise for high-precision measurements[2] (microscopy, spectroscopy) and for noiseless image amplification[3].
- Squeezing due to plane wave mixing is easy to describe and (conceptually) easy to measure.
- Existing treatments of squeezing, however, are not suited to describe the complex quantum correlations produced in realistic experiments involving entanglement between many modes.



*The nonlinear susceptibility which produced this pattern[4] ought also to produce quantum correlations. In which component(s) of the pattern should one look for reduced quantum noise?*

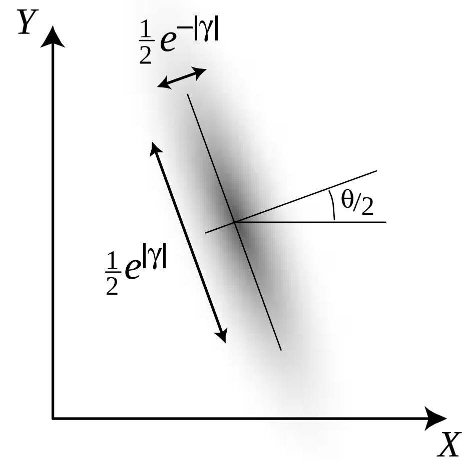
# The Goals

We wish to develop a general theory of multimode squeezing which:

- allows one to calculate any desired field correlation function
- describes how such correlations are affected by propagation (diffraction and linear optical processing)
- identifies which measurements of the field will reveal maximum quantum noise reduction
- provides physical insight!

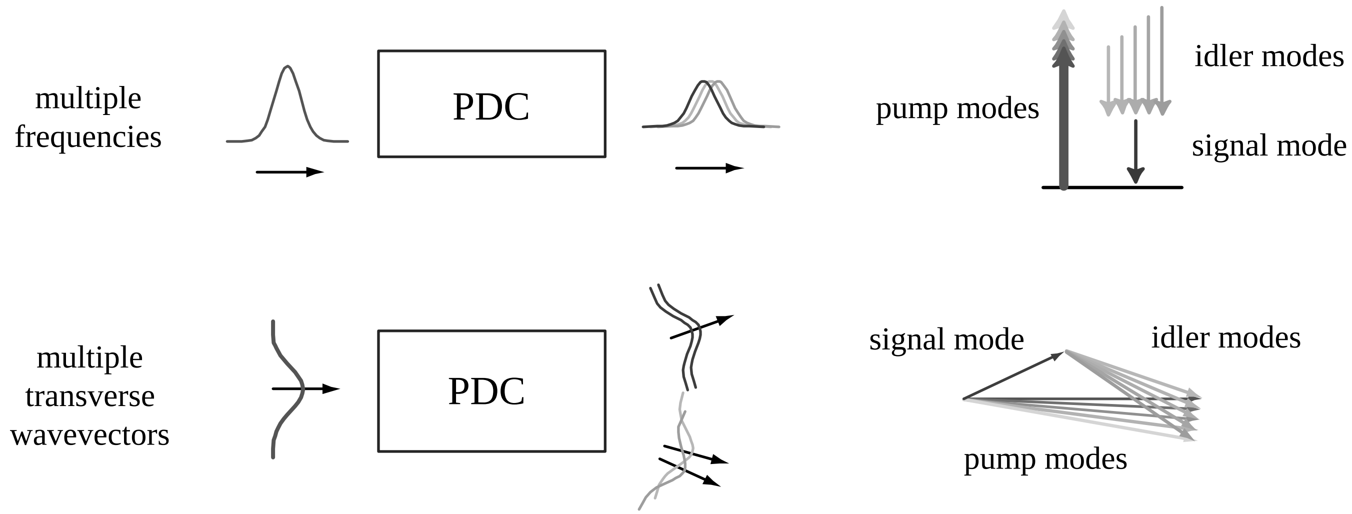
# Review: single-mode squeezing[5]

- Electric field operator $\hat{E} = \hat{a}e^{i(kz-\omega t)} + \hat{a}^{\dagger}e^{i(kz-\omega t)}$
- mode operator $\hat{a} = \hat{X} + i\hat{Y}$
- Squeeze operator $\hat{S} = \exp\left[\tfrac{1}{2}(\gamma^* \hat{a}\hat{a} - \gamma\hat{a}^{\dagger}\hat{a}^{\dagger})\right]$
- Squeeze parameter (= net parametric gain) $\gamma = |\gamma|e^{i\theta}$
- squeezed mode operator
$$\hat{a}_{\text{sqz}} \equiv \hat{S}^{\dagger}\hat{a}\hat{S} = \cosh|\gamma|\ \hat{a} - \sinh|\gamma|e^{i\theta}\ \hat{a}^{\dagger}$$
- Squeezed ($X_\theta$) and anti-squeezed ($Y_\theta$) quadrature operators $\hat{X}_\theta + i\hat{Y}_\theta = \hat{a}_{\text{sqz}}\, e^{-i\theta/2}$
- Quadrature variances $\Delta X_\theta = \tfrac{1}{2}e^{-|\gamma|}$ $\quad \Delta Y_\theta = \tfrac{1}{2}e^{|\gamma|}$



# Multimode squeezing: pictorial description

- When performing parametric downconversion with a pump having multiple frequencies and/or wavevectors, each signal(idler) mode becomes partially entangled with multiple idler(signal) modes via 2-photon emission:




- Do these distributed quantum correlations result in any measureable field component having reduced quantum noise?

# Multimode squeezing: mathematical description[6]

- Multimode squeeze operator (for any set of modes—spatial, spectral, waveguide)

$$\hat{S} = \exp\left[\tfrac{1}{2}\sum_{jk}(\Gamma^{*}_{jk}\hat{a}_j\hat{a}_k - \Gamma_{jk}\hat{a}^{\dagger}_j\hat{a}^{\dagger}_k)\right]$$

- squeezed mode operators

$$\begin{pmatrix}\hat{a}_1\\ \vdots\\ \hat{a}_n\end{pmatrix}_{\mathrm{sqz}} \equiv \hat{S}^{\dagger}\begin{pmatrix}\hat{a}_1\\ \vdots\\ \hat{a}_n\end{pmatrix}\hat{S}$$

$$= \cosh\tilde{\Gamma}\begin{pmatrix}\hat{a}_1\\ \vdots\\ \hat{a}_n\end{pmatrix} - \sinh\tilde{\Gamma}\,\exp(i\tilde{\Theta})\begin{pmatrix}\hat{a}_1\\ \vdots\\ \hat{a}_n\end{pmatrix}$$

where

$$\tilde{\Gamma}\,\exp(i\tilde{\Theta}) = \Gamma$$

- The squeeze matrix $\Gamma$ is analagous to the single-mode squeeze parameter $\gamma$. $\Gamma_{jk}$ is the net parametric gain of mode $j$ with mode $k$.
- The noise of any mode depends in a complicated way on many elements of $\Gamma$.
- No insight is gained!

# Eigenmode decomposition of the squeeze matrix

- The squeezing matrix can be diagonalized by a unitary transformation:

$$\boldsymbol{U}^{\dagger}\Gamma\,\boldsymbol{U}^{*}=\begin{pmatrix}\gamma_1 & & \\ & \ddots & \\ & & \gamma_n\end{pmatrix}$$

- Since the transformation is unitary, this corresponds to a physical change of basis for the field:

$$\begin{pmatrix}\hat{a}_1\\ \vdots \\ \hat{a}_n\end{pmatrix}\rightarrow \boldsymbol{U}\begin{pmatrix}\hat{a}_1\\ \vdots \\ \hat{a}_n\end{pmatrix}$$

- These field modes are the eigenmodes of the squeezing.

# Multimode squeezing in the eigenbasis

- In the eigenbasis of the squeezing, the expression for the squeezed mode operators reduces to

$$\begin{pmatrix} \hat{a}_1 \\ \vdots \\ \hat{a}_n \end{pmatrix}_{\text{sqz}} = \begin{pmatrix} \cosh\gamma_1\,\hat{a}_1 - \sinh\gamma_1\,\hat{a}_1^\dagger \\ \vdots \\ \cosh\gamma_n\,\hat{a}_n - \sinh\gamma_n\,\hat{a}_n^\dagger \end{pmatrix}$$

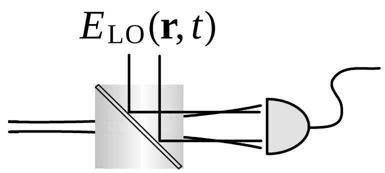
- Each eigenmode of the squeezing has the statistics of a single squeezed mode.

The eigenvectors of the squeezing matrix are *the* squeezed modes of the field, and the eigenvalues are the corresponding squeeze parameters.

# Eigenmodes and Measurement

- The photocurrent in homodyne detection is simply expressed in terms of the squeezing eigenvalues and the overlap between the local oscillator (LO) and each eigenmode.

$$\hat{a}(\mathbf{r},t)=\sum_j \phi_j(\mathbf{r},t)\hat{a}_{j,\mathrm{sqz}}$$




spatiotemporal profile of the $j$th eigenmode of the squeezing

- Overlap of the local oscillator with the $p$th eigenmode

$$O_p=\frac{\int E_{LO}^{*}\phi_p\, d\mathbf{r}\, dt}{\sqrt{\int |E_{LO}|^2 d\mathbf{r}\, dt}}$$

- photocurrent noise

$$\langle \Delta i^2\rangle=\sum_p \left|O_p\right|^2\left[e^{-2\gamma_p}\cos^2(\arg O_p)+e^{2\gamma_p}\sin^2(\arg O_p)\right]$$

The maximum squeezing is observed when the mode of the local oscillator is matched to that of the eigenmode having the largest eigenvalue.

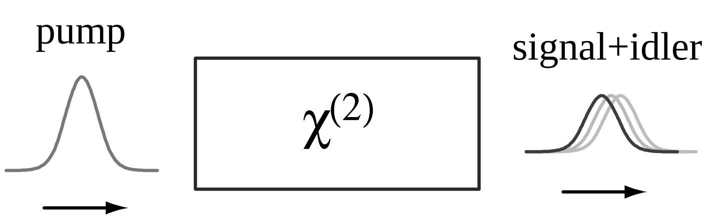
# Properties of the eigenmodes/eigenvalues of squeezing

- The eigenmodes are the optimal basis in which to measure quantum noise reduction
- The number of non-zero eigenvalues is the number of squeezed modes (pixels, channels) available for quantum imaging or communication
- The distribution of eigenvalues is unchanged by
  - lossless (paraxial) diffraction
  - lossless beam splitting
  - passage through lossless refractive or diffractive optics
- The statistics of any field mode can be simply expressed in terms of the squeezing eigenvalues

The number of squeezed modes and their degrees of squeezing are a fundamental, invariant (under lossless linear manipulation) property of the field.

# Spectrally multimode squeezing in a degenerate OPA/OPG

- Consider an OPA/OPG pumped by a short (spectrally broad) pulse:




$$\hat{H} = \frac{i\hbar c}{2} \sum_{jk} G_{jk}^{*} \hat{a}_j \hat{a}_k \exp(-i\Delta k_{jk} z) + \text{ H.c.}$$

$G_{jk}$ = parametric gain coefficient for mode pair $j, k$

$$\Delta k_{jk} = k(\omega_j + \omega_k) - k(\omega_j) - k(\omega_k)$$
$$= \text{phase mismatch}$$

- Propagation equation $\quad \frac{d}{dz}\hat{a}_j = \frac{i}{\hbar c}\left[\hat{H}, \hat{a}_j\right]$

- Solution $\quad \hat{a}(L) = \hat{S}^{\dagger} \hat{a}_j(0) \hat{S}$

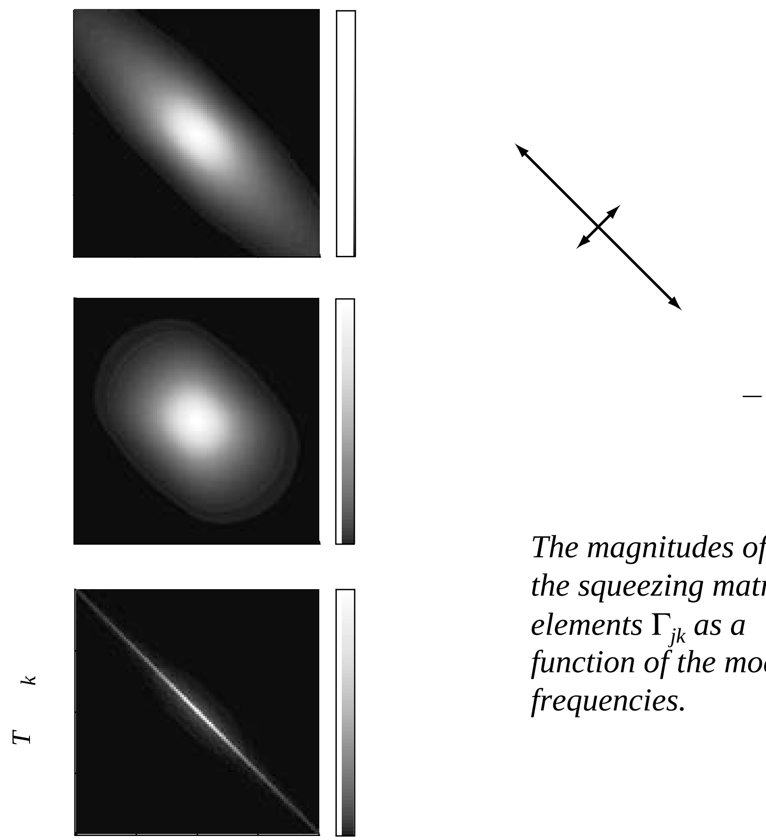
where $S$ is a multimode squeeze operator with $\Gamma$ determined by $\boldsymbol{G}$ and the degree of phase mismatch

# The squeezing matrix (net parametric gain) for pulse-pumped OPA/OPG

- With no phase mismatch, the squeezing matrix (net parametric gain) is just ***GL***

- Phase velocity mismatch reduces the effective bandwidth of the nonlinear response

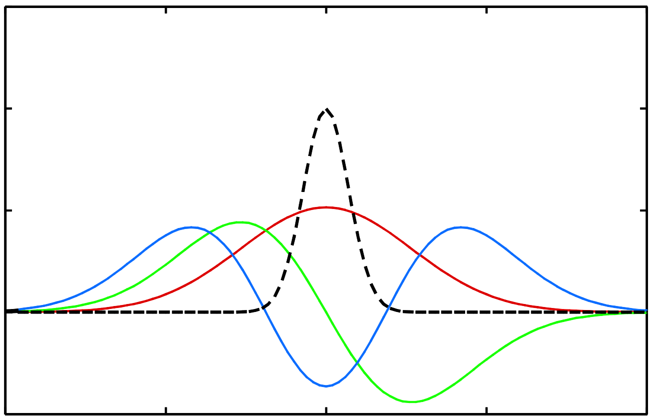
- Group velocity mismatch reduces the effective bandwidth of the pump



*The magnitudes of the squeezing matrix elements $\Gamma_{jk}$ as a function of the mode frequencies.*

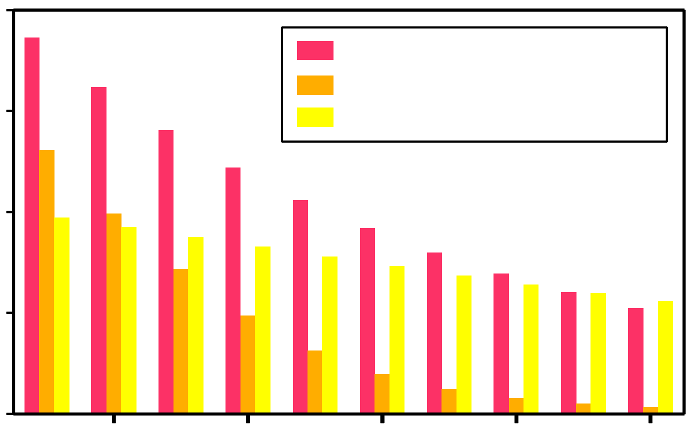
# The eigenmodes of squeezing in a pulse-pumped OPA/OPG

- The spectral amplitudes of the eigenmodes are (depending on the impulse response function of the nonlinearity) approximately Hermite-Gauss functions



*The spectral amplitudes of the first three eigenmodes of the squeezing (solid lines) and of a pulse derived from the pump (dashed line). $\Delta k$=0 and $\Delta\omega T_{\chi}$=0.1.*

# The eigenvalues of squeezing in a pulse-pumped OPA/OPG



*The ten largest eigenvalues of the squeezing for a pump pulse long compared to the nonlinear response time ($\Delta\omega T_\chi$=0.1) and with an intensity such that the gain-length product at the peak of the pump is 4. For the case of phase velocity mismatch, $\Delta kL$ was chosen to be $2\pi$. For the case of group velocity mismatch, the pump and downconverted fields were given a difference in group delay of $60T_\chi$.*

- With perfect phasematching, the squeeze parameter of the most-squeezed eigenmode is nearly equal to the gain-length product at the peak of the pump
- Both phase and velocity mismatch reduce the maximum degree of squeezing
- Phase velocity mismatch tends to reduce the effective number of squeezed modes
- Group velocity mismatch tends to increase the effective number of squeezed modes

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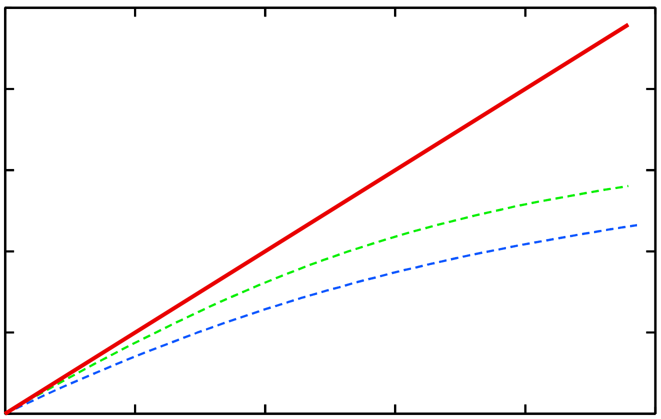
# Measured squeezing is improved significantly by matching the LO to the first eigenmode




*Comparison between the measureable amounts of squeezing in homodyne detection when the local oscillator is matched to first eigenmode (red) and to a pulse harmonically related to the pump (blue, green). Blue: $\Delta\omega T_\chi$=0.1; green: $\Delta\omega T_\chi$=0.5.*

- Matching the LO to the first eigenmode allows for the full squeezing present in the field to be observed, whereas a “common sense” choice for the LO results in many dB less observed squeezing.

Matching the LO to the eigenmodes of the squeezing can improve the measured amount of squeezing significantly

## Summary

- A general theory of multimode squeezing based on an eigenmode description was developed
- The theory is widely applicable: PDC in bulk media, waveguides, spatial and/or temporal domains
- The eigenmodes of the squeezing define a basis for the field which is
  - unique
  - physical
  - optimal for measurement of reduced quantum noise
- Any field correlation is easily calculated from the eigenvalues of the squeezing
- The number of squeezed modes and their degrees of squeezing are invariant under diffraction and lossless linear optical processing
- Knowledge of the squeeze eigenmodes allows a measurement of squeezing which is several dB better than possible with “common sense” mode-matching strategies

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