Influence of atmospheric turbulence on the propagation of quantum states of light carrying orbital angular momentum

Glenn A. Tyler[1] and Robert W. Boyd[2,*]

[1]*The Optical Sciences Company, P.O. Box 25309, Anaheim, California 92825, USA*
[2]*Department of Physics and Astronomy, The Institute of Optics, University of Rochester, Rochester, New York 14627, USA*
Corresponding author: boyd@optics.rochester.edu

Received August 8, 2008; accepted September 16, 2008;
posted November 6, 2008 (Doc. ID 99958); published January 12, 2009

We analyze the influence of atmospheric turbulence on the propagation of an optical vortex beam having the form $V(r,\theta)=A_0 e^{im\theta}$. The probability that a detected photon after propagating through the atmosphere has the same value of the orbital angular momentum as the launched photon is found to be given by $\langle s_0\rangle=[1+(1.845D/r_0)^2]^{-1/2}$, where D is the aperture diameter and r_0 is the Fried coherence diameter. These vortex beams behave very similarly to Laguerre–Gauss beams under the influence of atmospheric turbulence. These results have important implications for atmospheric laser communication systems that employ quantum encryption. © 2009 Optical Society of America

OCIS codes: 010.1330, 270.5585, 270.5565.

Several proposals have recently been made to use the orbital angular momentum (OAM) states [1,2] of light as a basis set for impressing quantum information onto single-photon light fields [3–5]. A key motivation for this idea is that the OAM states provide an infinite basis set for describing the transverse structure of the beam. In contrast, the often-used polarization degree of freedom provides only a two-dimensional Hilbert space. Thus the Hilbert space accessible using OAM states can be extremely large, with important consequences for quantum communications and quantum information processing.

One specific proposed application of OAM states is in the context of secure communication. In this scenario, quantum information is encoded onto a single photon, which might be a member of an entangled biphoton pair, and this photon is then transmitted through a free-space communication link to a distant receiver as shown schematically in Fig. 1. The quantum state of this photon is measured at the receiver, and secure communications are achieved by means of one of the standard protocols of quantum key distribution [6], appropriately modified to make use of the extended Hilbert space.

The integrity of such a communication system could be badly compromised if the OAM states became scrambled by atmospheric turbulence. For example, if a photon containing $m\hbar$ units of OAM were transmitted, but as a result of atmospheric turbulence the received photon is measured to carry OAM of $n\hbar$ with $n\neq m$, the quality of the communications channel would be degraded. In this Letter, we calculate, as a function of the strength of the atmospheric turbulence, the probability that the received photon carries an OAM different from that of the transmitted photon. The results of such a calculation provide quantitative predictions for the integrity of free-space quantum communication systems. Related calculations [7–9] considered OAM states in the form of Laguerre–Gauss (LG) modes. Our work differs in that we consider the case of a pure vortex beam, that is, a beam with uniform amplitude within the transmitting aperture. We treat this case both because it offers the opportunity for comparison with LG states and because pure vortex beams are likely to prove useful in practice, because these take advantage of the full aperture of the transmitter.

We assume that the transmitted field at the transmitter can be represented as

$$A(\mathbf{r})=A_0 W(r/R)e^{im\theta}, \tag{1}$$

where A_0 is the (spatially uniform) field amplitude, $W(x)$ is the aperture function defined so that $W(x)=1$ for $|x|\leq 1$ and zero otherwise, r and θ are the radial and azimuthal coordinates, and m is the OAM quantum number. We further assume that the field at the receiver aperture can be represented as

$$V(\mathbf{r})=A_0 W(r/R)e^{im\theta}e^{i\phi(\mathbf{r})}, \tag{2}$$

where $\phi(\mathbf{r})$ represents the turbulence-induced wavefront distortion at the receiver.

We next consider explicitly the nature of the angular momentum scrambling induced by atmospheric turbulence. We note that we can expand the quantity $\exp[i\phi(\mathbf{r})]$ in an azimuthal Fourier series as

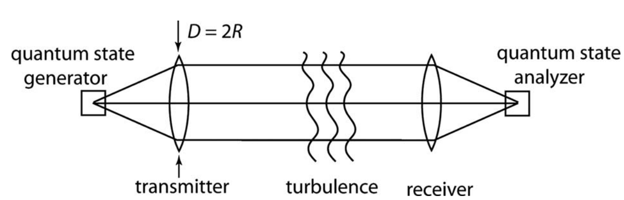

Fig. 1. Schematic of a free-space quantum communication link.

0146-9592/09/020142-3/$15.00

$$e^{i\phi(r,\theta)} = \sum_{l=-\infty}^{\infty} g_l(r) e^{il\theta}, \tag{3}$$

where the expansion coefficients $g_l(r)$ are given by

$$g_l(r) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}^{i\phi(r,\theta)} e^{-il\theta}. \tag{4}$$

Similarly, we expand the received field $V(\mathbf{r})$ in an azimuthal Fourier series as $V(r,\theta)=\Sigma_{l=-\infty}^{\infty} V_n(r)\exp(in\theta)$, where each Fourier component $V_n(r)$ is given by

$$V_n(r) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\theta V(r,\theta) e^{-in\theta}. \tag{5}$$

Equations (2) and (3) are now substituted into Eq. (5), which becomes

$$V_n(r) = \frac{A_0}{2\pi} W(r/R) \sum_{l=-\infty}^{\infty} g_l(r) \int_0^{2\pi} \mathrm{d}\theta e^{-i(n-l-m)\theta}. \tag{6}$$

The integral in this expression is equal to 2π if $n-l-m=0$ and vanishes otherwise. Using this result, the summation can be performed directly to give

$$V_n(r) = A_0 W(r/R) g_\Delta(r), \tag{7}$$

where we have defined Δ as $\Delta = n-m$. This result illustrates the manner in which the azimuthal Fourier components $g_\Delta(r)$ associated with atmospheric turbulence are coupled to the angular momentum state of the received field. It shows, for instance, that the spatial dc component (i.e., $\Delta=0$) of the azimuthal Fourier spectrum of $e^{i\phi(\mathbf{r})}$ is associated with the amount of radiation that remains in the initial OAM state.

Under many practical situations, one is interested primarily in determining the power contained in each OAM state of the received field. The total power collected by the receiver is given by

$$P = \tfrac{1}{2}\epsilon_0 c \int \mathrm{d}\mathbf{r} W(r/R) V^*(\mathbf{r}) V(\mathbf{r}) = \tfrac{1}{2}\epsilon_0 c |A_0|^2 \pi R^2, \tag{8}$$

where in obtaining the last form we used the field of Eq. (2). This power is distributed among the various (orthogonal) OAM modes of the field according to

$$P = \sum_{\Delta=-\infty}^{\infty} P_\Delta, \quad \text{where } P_\Delta = 2\pi |A_0|^2 \int_0^R \mathrm{d}r r g_\Delta^*(r) g_\Delta(r). \tag{9}$$

It is useful to consider the fraction $s_\Delta = P_\Delta / P$ of the power contained in each OAM mode given by

$$s_\Delta = \frac{2}{R^2}\int_0^R \mathrm{d}r r g_\Delta^*(r) g_\Delta(r). \tag{10}$$

For any statistical realization of the atmospheric turbulence, s_Δ gives the probability that the OAM quantum number n of the received photon departs from that m of the transmitted photon by the amount $\Delta = n-m$.

The result presented in Eq. (10) is valid for any realization of atmospheric turbulence. Usually we are interested in the ensemble average of this quantity, which is given by an equation of the same form with s_Δ replaced by $\langle s_\Delta\rangle$ and with $g_\Delta^*(r) g_\Delta(r)$ replaced by $\langle g_\Delta^*(r) g_\Delta(r)\rangle$, where the angle brackets $\langle\ldots\rangle$ represent an ensemble average over the turbulence statistics. To proceed, Eq. (4) is used to express Eq. (10) in terms of the random phase associated with atmospheric turbulence. One obtains

$$\langle s_\Delta\rangle = K\int_0^R \mathrm{d}r r \int_0^{2\pi} \mathrm{d}\theta_1 \int_0^{2\pi} \mathrm{d}\theta_2 \langle e^{-i[\phi(r,\theta_1)-\phi(r,\theta_2)]}\rangle e^{i\Delta(\theta_1-\theta_2)}, \tag{11}$$

where $K=1/(2\pi^2 R^2)$. The analysis proceeds using standard methods. Since the aberrations introduced by atmospheric turbulence are normal random variables, the ensemble average present in Eq. (11) can be expressed as

$$\langle e^{-i[\phi(r,\theta_1)-\phi(r,\theta_2)]}\rangle = e^{-1/2\langle[\phi(r,\theta_1)-\phi(r,\theta_2)]^2\rangle}. \tag{12}$$

The quantity $\langle[\phi(r,\theta_1)-\phi(r,\theta_2)]^2\rangle$ is known as the phase structure function. It can be evaluated by means of the Kolmogorov turbulence theory to give the result

$$\langle[\phi(\mathbf{r}_1)-\phi(\mathbf{r}_2)]^2\rangle = 6.88\left|\frac{\mathbf{r}_1-\mathbf{r}_2}{r_0}\right|^{5/3}, \tag{13}$$

where r_0 is Fried's coherence diameter, which is a measure of the transverse distance scale over which refractive index correlations remain correlated. When Eqs. (12) and (13) are introduced into Eq. (11), the resulting integral simplifies dramatically. The result becomes

$$\langle s_\Delta\rangle = \frac{1}{\pi}\int_0^1 \mathrm{d}\rho\rho \int_0^{2\pi} \mathrm{d}\theta e^{-3.44(D/r_0)^{5/3}(\rho\sin\theta/2)^{5/3}} \cos\Delta\theta, \tag{14}$$

where we have introduced $\rho = r/R$. This integral can readily be evaluated numerically; some of the results of this procedure are shown below.

We have found that the integral of Eq. (14) can be evaluated analytically in the limiting cases of very small or very large collecting apertures. For a small receiver aperture ($D/r_0 \to 0$), we expand the exponent of Eq. (14) in a power series in D/r_0 and retain only the first two terms. The resulting integrals can be performed analytically. We find that

$$\langle s_\Delta\rangle = \begin{cases} 1 - 1.01\left(\dfrac{D}{r_0}\right)^{5/3} & \text{for } \Delta = 0 \\ 0.142\dfrac{\Gamma\left(\Delta-\frac{5}{6}\right)}{\Gamma\left(\Delta+\frac{11}{6}\right)}\left(\dfrac{D}{r_0}\right)^{5/3} & \text{otherwise} \end{cases}, \tag{15}$$

where $\Gamma(x)$ is the usual gamma function. This expression shows how energy is lost from the transmitted OAM mode and is gained by the other modes as the strength of the turbulence increases. The integral can also be evaluated analytically in the opposite

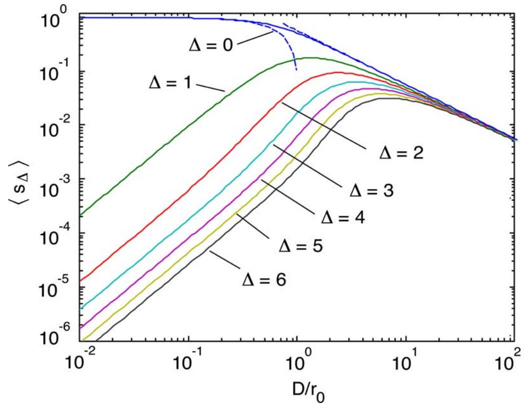

Fig. 2. (Color online) Quantity $\langle s_\Delta \rangle$ plotted against the strength of the atmospheric turbulence as quantified by the ratio of the telescope diameter D to the Fried parameter r_0 for several values of Δ; $\langle s_\Delta \rangle$ is the ensemble average of the fraction of the received power that is found to be in OAM mode $n=m+\Delta$, assuming that the transmitted beam was in OAM mode m. Solid curves give the predictions based on a numerical evaluation of the integral in Eq. (14). The dashed curves, shown only for $\Delta=0$, give the predictions of the asymptotic expressions of Eqs. (15) and (16).

limit of a very large aperture with $D/r_0 \gg 1$. We find that

$$\langle s_\Delta \rangle = \frac{12\Gamma(3/5)}{5\pi(3.44)^{3/5}}\left(\frac{D}{r_0}\right)^{-1} = 0.542\left(\frac{D}{r_0}\right)^{-1}. \quad (16)$$

This result shows that in the presence of strong turbulence all of the OAM states are populated with equal probability. Stated differently, all of the information content of the transmitted field is lost.

The key results of our analysis are presented in Fig. 2, in which $\langle s_\Delta \rangle$ is plotted as a function of D/r_0 for several values of Δ. The solid curve for $\Delta=0$ begins at unity and falls asymptotically as the strength of the turbulence increases. This curve thus provides a quantitative prediction for how quickly energy is lost from the mode that is transmitted. The curves for all other values of Δ initially increase with increasing turbulence levels and eventually decrease with further increases. The decrease at high turbulence levels occurs because the optical power is being spread among more and more OAM modes. Also shown in Fig. 2 as dashed curves are the prediction of the asymptotic expressions of Eqs. (15) and (16). To avoid cluttering Fig. 2, these results are shown only for $\Delta=0$. These approximate forms are extremely good in their respective limits. We have found that we can piece these forms together to obtain a formula for $\langle s_0 \rangle$ that is highly accurate over the entire domain of D/r_0. We obtain this formula by arguing that the inverse of $\langle s_0 \rangle$ should equal the square root of the sum of the squares of the inverses of the two asymptotic forms. In doing so, we take $\langle s_0 \rangle$ in the limit of small D/r_0 to be equal to unity. This procedure thus leads to the simple expression

$$\langle s_0 \rangle = [1 + (1.845D/r_0)^2]^{-1/2}. \quad (17)$$

This expression is remarkably accurate at predicting the value of the integral in Eq. (14). We find that there is at most a 0.1% difference between the value of $\langle s_0 \rangle$ given by numerical integration and that predicted by Eq. (17).

It is instructive to compare our results with those of Fig. 3 of Paterson in [7]. As mentioned above, Paterson treats a somewhat different problem, that of an OAM beam in the form of an LG function. Nonetheless, the two calculations lead to very similar predictions, with the one exception that Paterson's curves peak at a considerably smaller value of the abscissa as they are plotted against b/r_0, whereas ours are plotted against D/r_0. We conclude that LG beams and pure vortex beams perform essentially equivalently in terms of robustness against atmospheric turbulence, although vortex beams may offer some benefit in that they are often easier to create.

In summary, we present a calculation that quantifies the rate at which quantum information encoded on the OAM states of individual photons is lost as a result of propagation through atmospheric turbulence. These results are summarized by the simple relation of Eq. (17). These results should prove useful in the design of practical free-space quantum communication systems.

We acknowledge useful discussions with M. Gruneisen and W. Miller and support by an United States Air Force Office of Scientific Research (USAFOSR) grant.

References

1. J. Leach, J. Courtial, K. Skeldon, S. M. Barnett, S. Franke-Arnold, and M. J. Padgett, Phys. Rev. Lett. **92**, 013601 (2004).
2. G. Gibson, J. Courtial, M. J. Padgett, M. Vasnetsov, V. Pas'ko, S. M. Barnett, and S. Franke-Arnold, Opt. Express **12**, 5448 (2004).
3. A. Mair, A. Vaziri, G. Weihs, and A. Zeilinger, Nature **412**, 313 (2001).
4. G. Molina-Terriza, J. P. Torres, and L. Torner, Phys. Rev. Lett. **88**, 013601 (2002).
5. M. T. Gruneisen, W. A. Miller, R. C. Dymale, and A. M. Sweiti, Appl. Opt. **47**, A33 (2008).
6. N. Gisin and R. Thew, Nat. Photonics **1**, 165 (2007).
7. C. Paterson, Phys. Rev. Lett. **94**, 153901 (2005).
8. C. Gopaul and R. Andrews, New J. Phys. **9**, 94 (2007).
9. G. Gbur and R. K. Tyson, J. Opt. Soc. Am. A **25**, 255 (2008).