ECE440 - Introduction to Random Processes

Midterm Exam

October 30, 2024

Instructions:

- This is an open book, open notes exam.
- Calculators are not needed; laptops, tablets and cell-phones are not allowed.
- Perfect score: 100 (out of 101, extra point is a bonus point).
- Duration: 90 minutes.
- This exam has 10 numbered pages, check now that all pages are present.
- Make sure you write your name in the space provided below.
- Show all your work, and write your final answers in the boxes when provided.

Name:__________________________________

Problem	Max. Points	Score	Problem	Max. Points	Score
1.	16		5.	10	
2.	8		6.	12	
3.	13		7.	30	
4.	12				
			Total	101	

GOOD LUCK!

1. Suppose that $X_{\mathbb{N}} = X_0, X_1, \dots, X_n, \dots$ is a Markov chain with state space $S = \{1, 2\}$ and transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 1/6 & 5/6 \\ 2/3 & 1/3 \end{pmatrix}.$$

(a) (12 points) Compute the stationary distribution of $X_{\mathbb{N}}$.

(b) (4 points) Suppose that X_0 has the distribution obtained in part (a). $\mathbb{E}\left[X_2\right] = ?$

2. (a) (3 points) We say that events A and B where $\mathrm{P}(A) > 0$ and $\mathrm{P}(B) > 0$ are positively correlated if

$$\mathrm{P}\left(A \,\middle|\, B\right) > \mathrm{P}(A).$$

Prove or disprove that if A and B are positively correlated, then the following inequality holds:

$$\mathrm{P}\left(B \,\middle|\, A\right) > \mathrm{P}(B).$$

(b) (5 points) Let $\mathbb{I}\{A\}$ and $\mathbb{I}\{B\}$ be indicator random variables of events A and B in part (a), where

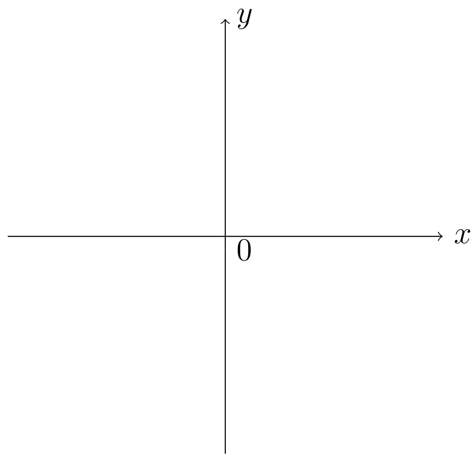
$$\mathbb{I}\{A\} = \begin{cases} 1, & \text{if event } A \text{ occurs,} \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \mathbb{I}\{B\} = \begin{cases} 1, & \text{if event } B \text{ occurs,} \\ 0, & \text{otherwise} \end{cases}.$$

Prove that $\mathrm{Cov}[\mathbb{I}\{A\}, \mathbb{I}\{B\}] > 0$ if and only if events A and B are positively correlated.

3. Consider the continuous random variables X and Y with joint probability density function

$$f_{XY}(x, y) = \begin{cases} e^{-x}, & 0 \leq y \leq x, \\ 0, & \text{otherwise.} \end{cases}$$

(a) (2 points) Sketch the region of $(x, y) \in \mathbb{R}^2$ where $f_{XY}(x, y)$ is non-zero.



(b) (3 points) Find the marginal probability density function $f_X(x)$.

(c) (4 points) Find the conditional probability density function $f_{Y|X}(y \mid x)$, where $x > 0$.

(d) (4 points) $\mathbb{E}\left[Y \mid X = 2\right] =?$

4. Consider a Markov chain $X_{\mathbb{N}} = X_0, X_1, \ldots, X_n, \ldots$ with state space $S = \{1, 2, 3, 4, 5, 6\}$ and transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 1/3 & 0 & 1/3 & 0 & 0 & 1/3 \\ 1/2 & 1/4 & 1/4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1/4 & 1/4 & 1/4 & 0 & 0 & 1/4 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

(a) (6 points) Draw the corresponding state transition diagram.

(b) (6 points) Specificy the communication classes and determine whether they are transient or recurrent.

5. Let $X_{\mathbb{N}} = X_1, X_2, \ldots, X_n, \ldots$ be an i.i.d. sequence of Poisson(2) random variables.

(a) (6 points) Consider the sample mean

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i.$$

How large should n be so that P $\left(|\bar{X}_n - 2| \geq 0.1\right) \leq 10^{-3}$? [Hint: Use Chebyshev's inequality]

(b) (4 points) Calculate

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} X_i^2$$

and provide justification for the existence of the limit.

6. (a) (4 points) Consider a standard Normal random variable $Z \sim \mathcal{N}(0,1)$. Compute $\mathrm{P}\left(-1 < Z < 1\right)$ and write your result in terms of the cumulative distribution function

$$\Phi(z) = \mathrm{P}\left(Z \leq z\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-u^2/2} du.$$

(b) (8 points) Suppose that we are trying to transmit a signal over a communication channel. During the transmission, the channel introduces additive noise from 100 independent corruption sources. Each individual source produces an amount of noise that is Uniformly distributed between $a = -1$ and $b = 1$. If the total amount of noise is greater than 10 or less than -10, then the received signal is useless. Find the approximate probability that the absolute value of the total amount of noise from the 100 sources is less than 10, in which case the transmitted signal can be correctly decoded. Write your result in terms of Φ, and justify your approximations.

7. Consider a branching process model for the evolution of a population and let X_n be the number of individuals in generation n. Suppose the k-th individual in generation n creates $Q_{k,n+1}$ individuals in generation $n+1$, and that the $Q_{k,n}$ are i.i.d. across individuals and generations, and independent of X_0. Let $\mu = \mathbb{E}\,[Q_{k,n}] > 0$ and $\sigma^2 = \text{var}[Q_{k,n}]$. Under the preceding assumptions, $X_{\mathbb{N}} = X_0, X_1, \ldots, X_n, \ldots$ is a Markov chain with state space $S = \{0, 1, 2, \ldots\}$ for which

$$X_{n+1} = Q_{1,n+1} + \ldots + Q_{X_n,n+1} \quad \text{if } X_n > 0,$$

and $X_{n+1} = 0$ if $X_n = 0$. Let $M_n = \mathbb{E}\,[X_n]$ and $V_n = \text{var}[X_n]$. Throughout, assume that $X_0 = 1$.

(a) (6 points) Derive an expression for M_{n+1} in terms of M_n and μ.

(b) (10 points) Derive an expression for V_{n+1} in terms of V_n, M_n, μ and σ^2.

(c) (8 points) Prove that $M_n = \mu^n$ and that $V_n = \sigma^2 \mu^{n-1}(1 + \mu + \ldots + \mu^{n-1})$. Show your work.

[Hint: You can argue by mathematical induction. *Base case:* Show that the claim holds true for $n = 1$. *Inductive step:* Supposing the claim is true for n, then show it also holds for $n + 1$.]

(d) (6 points) $\lim_{n\to\infty} V_n =?$ Discuss the cases $\mu > 1$, $\mu = 1$, and $0 < \mu < 1$.