ECE440 - Introduction to Random Processes

Midterm Exam

October 30, 2024

Instructions:

- This is an open book, open notes exam.
- Calculators are not needed; laptops, tablets and cell-phones are not allowed.
- Perfect score: 100 (out of 101, extra point is a bonus point).
- Duration: 90 minutes.
- This exam has 10 numbered pages, check now that all pages are present.
- Make sure you write your name in the space provided below.
- Show all your work, and write your final answers in the boxes when provided.

Name: SOLUTIONS

Problem	Max. Points	Score	Problem	Max. Points	Score
1.	16		5.	10	
2.	8		6.	12	
3.	13		7.	30	
4.	12				
			Total	101	

GOOD LUCK!

1. Suppose that $X_{\mathbb{N}} = X_0, X_1, \ldots, X_n, \ldots$ is a Markov chain with state space $S = \{1, 2\}$ and transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 1/6 & 5/6 \\ 2/3 & 1/3 \end{pmatrix}.$$

(a) (12 points) Compute the stationary distribution of $X_{\mathbb{N}}$.

$$\boldsymbol{\pi} = \left[\frac{4}{9}, \frac{5}{9}\right]^\top$$

The unique stationary distribution $\boldsymbol{\pi} = [\pi_1, \pi_2]^\top$ (the Markov chain is ergodic) satisfies

$$\begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix} = \begin{pmatrix} 1/6 & 2/3 \\ 5/6 & 1/3 \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix}, \qquad \pi_1 + \pi_2 = 1.$$

Solving the linear system yields $\boldsymbol{\pi} = [4/9, 5/9]^\top$.

(b) (4 points) Suppose that X_0 has the distribution obtained in part (a). $\mathbb{E}[X_2] =?$

$$\frac{14}{9}$$

If the initial distribution is $\boldsymbol{\pi}$, then $\boldsymbol{\pi}$ will be the distribution for all subsequent time instants $n \geq 1$. Hence, the expectation is $\mathbb{E}[X_2] = 1 \times \frac{4}{9} + 2 \times \frac{5}{9} = \frac{14}{9}$.

2. (a) (3 points) We say that events A and B where $\mathrm{P}(A) > 0$ and $\mathrm{P}(B) > 0$ are positively correlated if

$$\mathrm{P}(A \mid B) > \mathrm{P}(A).$$

Prove or disprove that if A and B are positively correlated, then the following inequality holds:

$$\mathrm{P}(B \mid A) > \mathrm{P}(B).$$

From Bayes' rule and the positive correlation assumption it follows that

$$1 < \frac{\mathrm{P}(A \mid B)}{\mathrm{P}(A)} = \frac{\mathrm{P}(B \mid A)}{\mathrm{P}(B)},$$

which immediately implies $\mathrm{P}(B \mid A) > \mathrm{P}(B)$.

(b) (5 points) Let $\mathbb{I}\{A\}$ and $\mathbb{I}\{B\}$ be indicator random variables of events A and B in part (a), where

$$\mathbb{I}\{A\} = \begin{cases} 1, & \text{if event } A \text{ occurs,} \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \mathbb{I}\{B\} = \begin{cases} 1, & \text{if event } B \text{ occurs,} \\ 0, & \text{otherwise} \end{cases}.$$

Prove that $\mathrm{Cov}[\mathbb{I}\{A\}, \mathbb{I}\{B\}] > 0$ if and only if events A and B are positively correlated.

From the definition, the covariance of $\mathbb{I}\{A\}$ and $\mathbb{I}\{B\}$ is

$$\begin{aligned}\text{Cov}[\mathbb{I}\{A\},\mathbb{I}\{B\}] &= \mathbb{E}[\mathbb{I}\{A\}\times\mathbb{I}\{B\}] - \mathbb{E}[\mathbb{I}\{A\}]\,\mathbb{E}[\mathbb{I}\{B\}] \\ &= \mathrm{P}(A\cap B) - \mathrm{P}(A)\,\mathrm{P}(B),\end{aligned}$$

where we used $\mathbb{I}\{A\}\times\mathbb{I}\{B\} = \mathbb{I}\{A\cap B\}$ and that the expectation of an indicator random variable is the probability of the indicated event. Now, since $\mathrm{P}(A\cap B) = \mathrm{P}(A\,|\,B)\,\mathrm{P}(B)$ then we find

$$\begin{aligned}\text{Cov}[\mathbb{I}\{A\},\mathbb{I}\{B\}] &= \mathrm{P}(A\cap B) - \mathrm{P}(A)\,\mathrm{P}(B) \\ &= [\mathrm{P}(A\,|\,B) - \mathrm{P}(A)]\,\mathrm{P}(B),\end{aligned}$$

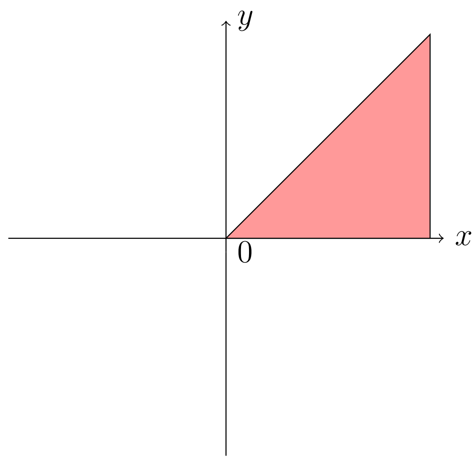
which will be strictly positive if and only if $\mathrm{P}(A\,|\,B) > \mathrm{P}(A)$ (recall we assume $\mathrm{P}(B) > 0$), meaning that A and B are positively correlated.

3. Consider the continuous random variables X and Y with joint probability density function

$$f_{XY}(x,y) = \begin{cases} e^{-x}, & 0\le y\le x, \\ 0, & \text{otherwise.}\end{cases}$$

(a) (2 points) Sketch the region of $(x,y)\in\mathbb{R}^2$ where $f_{XY}(x,y)$ is non-zero.

The support of $f_{XY}(x,y)$ corresponds to the region $(x,y)\in\mathbb{R}^2$ such that $0\le y\le x$.



(b) (3 points) Find the marginal probability density function $f_X(x)$.

$$\boxed{f_X(x) = \begin{cases} xe^{-x}, & x\ge 0, \\ 0, & \text{otherwise.}\end{cases}}$$

To obtain the marginal pdf $f_X(x)$ we integrate $f_{XY}(x,y)$ over all values of y for each x, namely

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)dy = \int_0^x e^{-x}dy = xe^{-x},\ \ x\ge 0.$$

Notice how the integration limits are given by the support $0\le y\le x$ identified in part (a).

(c) (4 points) Find the conditional probability density function $f_{Y|X}(y \mid x)$, where $x > 0$.

$$f_{Y|X}(y \mid x) = \begin{cases} \frac{1}{x}, & x > 0,\ 0 \le y \le x, \\ 0, & \text{otherwise.} \end{cases}$$

From the definition of conditional pdf, we have

$$f_{Y|X}(y \mid x) = \frac{f_{XY}(x, y)}{f_X(x)}.$$

Using the expression for the marginal pdf derived in part (b), we readily obtain the result

$$f_{Y|X}(y \mid x) = \begin{cases} \frac{1}{x}, & x > 0,\ 0 \le y \le x, \\ 0, & \text{otherwise.} \end{cases}$$

(d) (4 points) $\mathbb{E}\left[Y \mid X = 2\right] =$?

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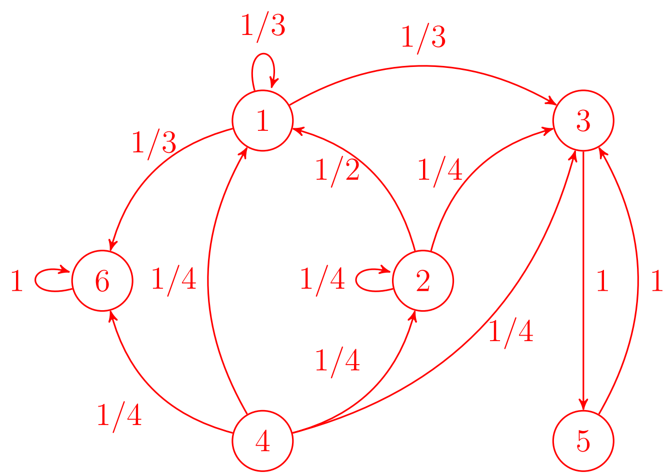
From the definition of conditional expectation for continuous random variables, we have

$$\mathbb{E}\left[Y \mid X = 2\right] = \int_{-\infty}^{\infty} y f_{Y|X}(y \mid 2) dy = \int_0^2 \frac{y}{2} dy = 1.$$

4. Consider a Markov chain $X_{\mathbb{N}} = X_0, X_1, \ldots, X_n, \ldots$ with state space $S = \{1, 2, 3, 4, 5, 6\}$ and transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 1/3 & 0 & 1/3 & 0 & 0 & 1/3 \\ 1/2 & 1/4 & 1/4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1/4 & 1/4 & 1/4 & 0 & 0 & 1/4 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

(a) (6 points) Draw the corresponding state transition diagram.



(b) (6 points) Specificy the communication classes and determine whether they are transient or recurrent.

State 6 is absorbing, hence it comprises its own recurrent communication class $\mathcal{R}_1 = \{6\}$. Similarly, states 3 and 5 communicate but no other state is accessible from them. Hence, they form a second recurrent communication class $\mathcal{R}_2 = \{3, 5\}$. State 1 only comunicates with itself, so it comprises its own transient class $\mathcal{T}_1 = \{1\}$ (being at state 1, there is a positive probability of being absorbed by classes $\mathcal{R}_1$ or $\mathcal{R}_2$). The same is true for states 2 and 4, which yield two additional transient classes $\mathcal{T}_2 = \{2\}$ and $\mathcal{T}_3 = \{4\}$.

5. Let $X_{\mathbb{N}} = X_1, X_2, \ldots, X_n, \ldots$ be an i.i.d. sequence of Poisson(2) random variables.

(a) (6 points) Consider the sample mean

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

How large should n be so that $\mathrm{P}\left(|\bar{X}_n - 2| \geq 0.1\right) \leq 10^{-3}$? [Hint: Use Chebyshev's inequality]

2×10^5

Recall that for $X_i \sim$ Poisson(2), then $\mathbb{E}\left[X_i\right] = 2$ and $\text{var}\left[X_i\right] = 2$. Now, for the sample mean $\bar{X}_n$ of n i.i.d. Poisson(2) random variables we have $\mathbb{E}\left[\bar{X}_n\right] = 2$ and $\text{var}\left[\bar{X}_n\right] = \frac{2}{n}$. From Chebyshev's inequality,

$$\mathrm{P}\left(|\bar{X}_n - 2| \geq 0.1\right) \leq \frac{\text{var}\left[\bar{X}_n\right]}{(0.1)^2} = \frac{2}{n \times 10^{-2}}.$$

For the probability in the right-hand-side to equal 10^{-3}, then we must have $n = 2 \times 10^5$.

(b) (4 points) Calculate

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} X_i^2$$

and provide justification for the existence of the limit.

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Because $X_{\mathbb{N}}$ is i.i.d., then $Y_{\mathbb{N}} = X_1^2, X_2^2, \ldots, X_n^2, \ldots$ is also i.i.d. By the strong law of large numbers the limit exists and is equal to

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} X_i^2 = \mathbb{E}\left[X_1^2\right] = \text{var}\left[X_1\right] + \left(\mathbb{E}\left[X_1\right]\right)^2 = 6, \quad \text{w.p. } 1.$$

6. (a) (4 points) Consider a standard Normal random variable $Z \sim \mathcal{N}(0,1)$. Compute $\text{P}\left(-1 < Z < 1\right)$ and write your result in terms of the cumulative distribution function

$$\Phi(z) = \text{P}\left(Z \leq z\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-u^2/2} du.$$

$2\Phi(1) - 1$

Using the definition of $\Phi(z)$ and the symmetry properties of the Normal pdf, we find

$$\begin{aligned}\text{P}\left(-1 < Z < 1\right) &= \text{P}\left(Z < 1\right) - \text{P}\left(Z < -1\right) \\ &= \text{P}\left(Z < 1\right) - \text{P}\left(Z > 1\right) \\ &= \text{P}\left(Z < 1\right) - \left(1 - \text{P}\left(Z < 1\right)\right) \\ &= 2\Phi(1) - 1.\end{aligned}$$

(b) (8 points) Suppose that we are trying to transmit a signal over a communication channel. During the transmission, the channel introduces additive noise from 100 independent corruption sources. Each individual source produces an amount of noise that is Uniformly distributed between $a = -1$ and $b = 1$. If the total amount of noise is greater than 10 or less than -10, then the received signal is useless. Find the approximate probability that the absolute value of the total amount of noise from the 100 sources is less than 10, in which case the transmitted signal can be correctly decoded. Write your result in terms of Φ, and justify your approximations.

$2\Phi(\sqrt{3}) - 1$

Let U_i, $i = 1, \ldots, 100$, be the i.i.d. Uniform$[-1,1]$ noise sources, with $\mathbb{E}\left[U_i\right] = 0$ and $\text{var}\left[U_i\right] = \mathbb{E}\left[U_i^2\right] = \frac{1}{3}$. Let $S_{100} = \sum_{i=1}^{100} U_i$ denote the total amount of noise that corrupts the transmitted signal. We are asked to approximate the probability $\text{P}\left(|S_{100}| \leq 10\right) = \text{P}\left(-10 \leq S_{100} \leq 10\right)$. By rescaling S_{100} and relying on the Central Limit Theorem, we obtain ($Z \sim \mathcal{N}(0,1)$ below)

$$\begin{aligned}\text{P}\left(-10 \leq S_{100} \leq 10\right) &= \text{P}\left(-\frac{10}{\sqrt{\text{var}\left[S_{100}\right]}} \leq \frac{S_{100}}{\sqrt{\text{var}\left[S_{100}\right]}} \leq \frac{10}{\sqrt{\text{var}\left[S_{100}\right]}}\right) \\ &= \text{P}\left(-\frac{10}{\sqrt{1/3 \times 100}} \leq \frac{S_{100}}{\sqrt{\text{var}\left[S_{100}\right]}} \leq \frac{10}{\sqrt{1/3 \times 100}}\right) \\ &\approx \text{P}\left(-\sqrt{3} \leq Z \leq \sqrt{3}\right) = 2\Phi(\sqrt{3}) - 1.\end{aligned}$$

7. Consider a branching process model for the evolution of a population and let X_n be the number of individuals in generation n. Suppose the k-th individual in generation n creates $Q_{k,n+1}$ individuals in generation $n+1$, and that the $Q_{k,n}$ are i.i.d. across individuals and generations, and independent of X_0. Let $\mu = \mathbb{E}[Q_{k,n}] > 0$ and $\sigma^2 = \text{var}[Q_{k,n}]$. Under the preceding assumptions, $X_{\mathbb{N}} = X_0, X_1, \ldots, X_n, \ldots$ is a Markov chain with state space $S = \{0, 1, 2, \ldots\}$ for which

$$X_{n+1} = Q_{1,n+1} + \ldots + Q_{X_n,n+1} \quad \text{if } X_n > 0,$$

and $X_{n+1} = 0$ if $X_n = 0$. Let $M_n = \mathbb{E}[X_n]$ and $V_n = \text{var}[X_n]$. Throughout, assume that $X_0 = 1$.

(a) (6 points) Derive an expression for M_{n+1} in terms of M_n and μ.

$$\boxed{M_{n+1} = \mu M_n}$$

The number of individuals in generation $n+1$ is given by the compound random variable

$$X_{n+1} = \sum_{k=1}^{X_n} Q_{k,n+1}.$$

To compute $M_{n+1} = \mathbb{E}[X_{n+1}]$ we condition on X_n. Because the $Q_{k,n+1}$ are i.i.d. we find that $\mathbb{E}\left[X_{n+1} \,\middle|\, X_n\right] = \mu X_n$. Hence, from the law of iterated expectations we obtain

$$M_{n+1} = \mathbb{E}[X_{n+1}] = \mathbb{E}\left[\mathbb{E}\left[X_{n+1} \,\middle|\, X_n\right]\right] = \mathbb{E}[\mu X_n] = \mu \mathbb{E}[X_n] = \mu M_n.$$

(b) (10 points) Derive an expression for V_{n+1} in terms of V_n, M_n, μ and σ^2.

$$\boxed{V_{n+1} = \sigma^2 M_n + \mu^2 V_n}$$

Likewise, to compute $V_{n+1} = \text{var}[X_{n+1}]$ we condition on X_n. Because the $Q_{k,n+1}$ are i.i.d. we find that $\text{var}\left[X_{n+1} \,\middle|\, X_n\right] = \sigma^2 X_n$. Using the conditional variance formula

$$\begin{aligned}
\text{var}[X_{n+1}] &= \mathbb{E}\left[\text{var}\left[X_{n+1} \,\middle|\, X_n\right]\right] + \text{var}\left[\mathbb{E}\left[X_{n+1} \,\middle|\, X_n\right]\right] \\
&= \mathbb{E}\left[\sigma^2 X_n\right] + \text{var}[\mu X_n] \\
&= \sigma^2 \mathbb{E}[X_n] + \mu^2 \text{var}[X_n] = \sigma^2 M_n + \mu^2 V_n.
\end{aligned}$$

(c) (8 points) Prove that $M_n = \mu^n$ and that $V_n = \sigma^2 \mu^{n-1}(1 + \mu + \ldots + \mu^{n-1})$. Show your work.

[Hint: You can argue by mathematical induction. *Base case:* Show that the claim holds true for $n = 1$. *Inductive step:* Supposing the claim is true for n, then show it also holds for $n + 1$.]

From the recursion $M_n = \mu M_{n-1}$ derived in part (a), it immediately follows that $M_n = \mu^n M_0$. But since $X_0 = 1$, then $M_0 = \mathbb{E}[X_0] = 1$ and so $M_n = \mu^n$ as desired.

From the variance recursion in part (b) and substituting the expression for M_{n-1} just derived, we have

$$V_n = \sigma^2 \mu^{n-1} + \mu^2 V_{n-1}.$$

We now proceed by mathematical induction. To establish the base case, note that $X_0 = 1$ and thus $V_0 = \text{var}[X_0] = 0$. So $V_1 = \sigma^2$ as desired. For the inductive step, we assume $V_n = \sigma^2 \mu^{n-1}(1 + \mu + \ldots + \mu^{n-1})$ holds and want to show $V_{n+1} = \sigma^2 \mu^n (1 + \mu + \ldots + \mu^n)$. To this end,

$$\begin{aligned} V_{n+1} &= \sigma^2 \mu^n + \mu^2 V_n \\ &= \sigma^2 \mu^n + \mu^2 \left(\sigma^2 \mu^{n-1}(1 + \mu + \ldots + \mu^{n-1})\right) \\ &= \sigma^2 \mu^n + \sigma^2 \mu^n \left(\mu + \mu^2 + \ldots + \mu^n\right) \\ &= \sigma^2 \mu^n (1 + \mu + \ldots + \mu^n) \end{aligned}$$

completing the proof.

(d) (6 points) $\lim_{n\to\infty} V_n = ?$ Discuss the cases $\mu > 1$, $\mu = 1$, and $0 < \mu < 1$.

Using the expression for V_n derived in part (c) it readily follows that

$$\lim_{n\to\infty} V_n = \begin{cases} 0, & 0 < \mu < 1, \\ \infty, & \mu \geq 1. \end{cases}$$