Performance Evaluation of Burst-Error-Correcting Codes on a Gilbert–Elliott Channel

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***Abstract*—In this letter the performance of single burst-error-correcting (BEC) codes used over bursty channels is evaluated. The channel is represented by the Gilbert–Elliott (GE) model, which has been used by numerous authors to evaluate the performance of random-error-correcting (REC) codes over bursty channels. Recursive expressions are derived, which are used in evaluating the probability of a codeword error. These expressions and an approximate closed-form expression are applied to the performance of a single (23,12) BEC code.**

***Index Terms*— Burst-error-correcting codes, cyclic codes, Gilbert–Elliott channel.**

I. Introduction

In many digital communication systems, symbol errors at the output of the channel tend to occur in bursts. The Gilbert–Elliott (GE) channel [1], [2] is a useful discrete model for such channels that has been studied in considerable detail in the literature. For Rayleigh-fading channels, several researchers have related the model parameters of the GE channel to the fade statistics [3]–[5], and such a representation has been shown to be fairly accurate in spite of its simplicity [6]. Elliott [2] first analyzed the performance of error-correcting codes on a GE channel by establishing a series of recursions for $P(m,n)$, the probability of m transmission errors in a block of n symbols. Recently, Yee and Weldon [7] presented a combinatorial analysis for a simplified GE channel that replaced the recursions with closed-form expressions. An alternate nonrecursive technique for approximate evaluation of $P(m,n)$ on simplified GE channels has also been presented by Wong and Leung [8].

Since the probability $P(m,n)$ takes into account only the total number of errors and disregards their distribution, it is useful only for the performance evaluation of random-error-correcting (REC) codes and cannot be used for burst-error-correcting (BEC) codes. An evaluation of two BEC codes on a GE channel was performed in [5] using Monte Carlo simulations, where a simplified analysis for the codeword error probability for BEC codes was also presented under rather restrictive assumptions on the model parameters. In this letter a modified analysis of the GE channel is presented for evaluating the performance of single BEC codes. Since these codes are particularly designed for combating burst errors, such an analysis is highly desirable.

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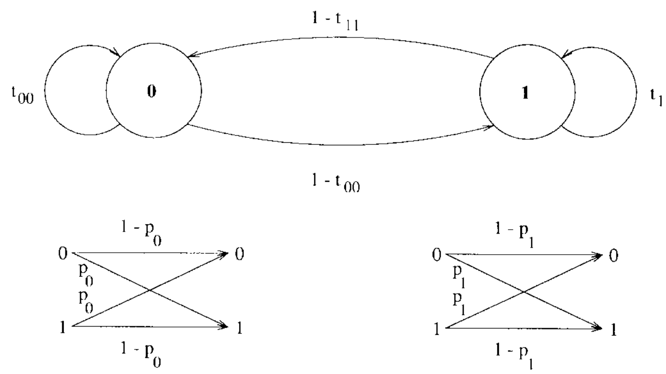

Fig. 1. The GE channel model.

II. GE Channel Model

The GE channel considered in this letter is a binary symmetric channel (BSC) with memory determined by a two-state Markov chain. A schematic of the model is shown in Fig. 1. The two states of the channel are $\mathbf{0}$ and $\mathbf{1}$, with the state transition probabilities $t_{ij} = \mathcal{P}(\text{state } n+1 \text{ is } j \mid \text{state } n \text{ is } i)$, where $\mathcal{P}(E)$ indicates the probability of event E and the $|$ separates a conditioning event. In state i the channel is a BSC with symbol-error probability p_i. Note that with this notation, $(1-i)$ denotes the state different from i, and $t_{ij} = 1 - t_{ii}$, $j \neq i$. In the subsequent analysis it will be assumed that the channel undergoes a state transition at the beginning of a symbol time and then the symbol is transmitted over the BSC determined by the state.

III. BEC Analysis

For the analysis of BEC capabilities, it is necessary to keep track of the channel state in addition to the number of errors. Let

$$P_{ij}(m,n) = \mathcal{P}(m \text{ errors in an } n \text{ symbol sequence and ending state is } j \mid \text{initial state was } i) \quad (1)$$

where "initial state" refers to the state for the symbol preceding the first symbol in the n symbol sequence and "ending state" refers to the state for the last symbol of the n symbol sequence.

The steady-state probability of the channel being in state i is $\pi_i = ((1 - t_{(1-i)(1-i)})/(1 - t_{00} + 1 - t_{11}))$ [9, pp.

349–354]. From Bayes' rule, it follows that

$$\begin{aligned} P_j(m,n) &\overset{\text{def}}{=} \mathcal{P}(m \text{ errors in an } n \text{ symbol} \\ &\qquad \text{sequence and ending state is } j) \\ &= \pi_0 P_{0j}(m,n) + \pi_1 P_{1j}(m,n), \qquad j = 0,1 \end{aligned} \tag{2}$$

$$\begin{aligned} P^i(m,n) &\overset{\text{def}}{=} \mathcal{P}(m \text{ errors in an } n \text{ symbol} \\ &\qquad \text{sequence} \mid \text{initial state is } i) \\ &= P_{i0}(m,n) + P_{i1}(m,n), \qquad i = 0,1. \end{aligned} \tag{3}$$

In terms of these definitions, the probability of m errors in a sequence of n symbols $P(m,n)$ can now be written as

$$P(m,n) = P_0(m,n) + P_1(m,n) = \pi_0 P^0(m,n) + \pi_1 P^1(m,n). \tag{4}$$

From the channel model, it can be readily seen that the following coupled recursions hold for $n = 0,1,2,3,\cdots$ and $m = 0,1,2,\cdots,n$:

$$\begin{aligned} &P_{ij}(m,n) \\ &\quad = P_{ij}(m,n-1)t_{jj}(1-p_j) + P_{i(1-j)}(m,n-1) \\ &\qquad \cdot (1-t_{(1-j)(1-j)})(1-p_j) + P_{ij}(m-1,n-1)t_{jj}p_j \\ &\qquad + P_{i(1-j)}(m-1,n-1)(1-t_{(1-j)(1-j)})p_j, \\ &\qquad\qquad i,j = 0,1. \end{aligned} \tag{5}$$

The initial conditions for the recursion are given by $P_{ij}(m,n) = 0 \forall m > n$ and $m < 0$, and

$$P_{ij}(0,0) = \begin{cases} 1, & i = j \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

In order to evaluate the performance of error-correcting codes, consider the use of a code with block length N on a GE channel. Clearly, if the code is a REC code capable of correcting up to t random errors, the probability of correct decoding is $\sum_{m=0}^{t} P(m,N)$. However, if the code is a BEC code, the probability of correct decoding cannot be expressed in terms of $P(m,n)$ alone. In order to analyze the performance of BEC codes their correction capabilities must be defined. Since a large number of useful BEC codes are cyclic codes capable of correcting "cyclic bursts," an analysis encompassing both cyclic and noncyclic BEC codes is considered.[1] The difference between the received vector and the corresponding transmitted codeword defines the error vector. A nonzero error vector is said to have a (cyclic) burst length p if all of the transmission errors are among p (cyclically) successive components, the first and last of which are nonzero [10]. The burst length of the zero error vector will be considered 0. A (cyclic) BEC code is said to have a (cyclic) burst-correcting capability (BCC) of t if it is capable of correcting all error vectors with (cyclic) length less than or equal to t. Similarly, an REC code will be said to have an REC capability (RCC) of t if it is capable of correcting up to t random errors.

[1] Note that practical considerations often necessitate the use of shortened cyclic codes that are not cyclic.

Given the (cyclic) BCC of a (cyclic) BEC code, the probability of correct decoding can readily be determined from

$$\begin{aligned} B(k,N) &\overset{\text{def}}{=} \mathcal{P}(\text{errors in a sequence of } N \text{ symbols} \\ &\qquad \text{have burst length} \le k) \end{aligned} \tag{7}$$

$$\begin{aligned} B^{\text{cyc}}(k,N) &\overset{\text{def}}{=} \mathcal{P}(\text{errors in a sequence of } N \text{ symbols} \\ &\qquad \text{have cyclic burst length} \le k). \end{aligned} \tag{8}$$

Observe that $B(k,N) = P(0,N) + \sum_{i=1}^{N} B_i(k,N)$, where

$$\begin{aligned} B_i(k,N) &\overset{\text{def}}{=} \mathcal{P}(1 \le \text{burst length} \le k \text{ and first error occurs} \\ &\qquad \text{in } i\text{th position in a sequence of } N \text{ symbols}). \end{aligned} \tag{9}$$

By considering all cases that lead to an error in the ith position, it can be seen that for $1 \le i < (N-k+1)$

$$\begin{aligned} B_i(k,N) &= \sum_{l=0}^{1}\sum_{j=0}^{1} P_j(0,i-1) \\ &\quad \cdot \left[t_{jj}p_j T_{jl}^{(k-1)} + (1-t_{jj})p_{(1-j)} T_{(1-j)l}^{(k-1)}\right] \\ &\quad \cdot P^l(0,N+1-(i+k)) \end{aligned} \tag{10}$$

and for $(N-k+1) \le i \le N$

$$B_i(k,N) = \sum_{j=0}^{1} P_j(0,i-1)[t_{jj}p_j + (1-t_{jj})p_{(1-j)}] \tag{11}$$

where

$$\begin{aligned} T_{pq}^{(l)} &\overset{\text{def}}{=} \mathcal{P}(\text{state } n+l \text{ is } q \mid \text{state } n \text{ is } p) \\ &= \sum_{m=0}^{l} P_{pq}(m,l). \end{aligned} \tag{12}$$

In an analogous fashion it can be seen that for $k < N/2+1$ (a condition that would hold for most reasonable cyclic BEC codes), $B^{\text{cyc}}(k,N) = P(0,N) + \sum_{i=1}^{N} B_i^{\text{cyc}}(k,N)$, where

$$\begin{aligned} B_i^{\text{cyc}}(k,N) &= B_i(k,N), \qquad 1 \le i \le (N-k+1) \\ B_i^{\text{cyc}}(k,N) &= \sum_{l=0}^{1}\sum_{j=0}^{1} \pi_j P_{jl}(0,N-k)[t_{ll}p_l + (1-t_{ll})p_{1-l}] \\ &= \sum_{l=0}^{1} P_l(0,N-k)[t_{ll}p_l + (1-t_{ll})p_{1-l}], \\ &\qquad (N-k+1) \le i \le N. \end{aligned} \tag{13}$$

IV. Simplified Analysis for Certain Channels

If the state transition probabilities t_{01} and t_{10} are small in comparison with $1/N$ (the inverse of the block length), an alternate analysis may be used to arrive at closed-form expressions for the codeword-error probabilities. Such situations often arise when the parameters of common communication channels are related to the transition probabilities of the GE model [5]. For these scenarios, the transitions between states are extremely infrequent and one may assume that for the duration of a codeword, the channel remains in the state in which the codeword began. With this approximation, the

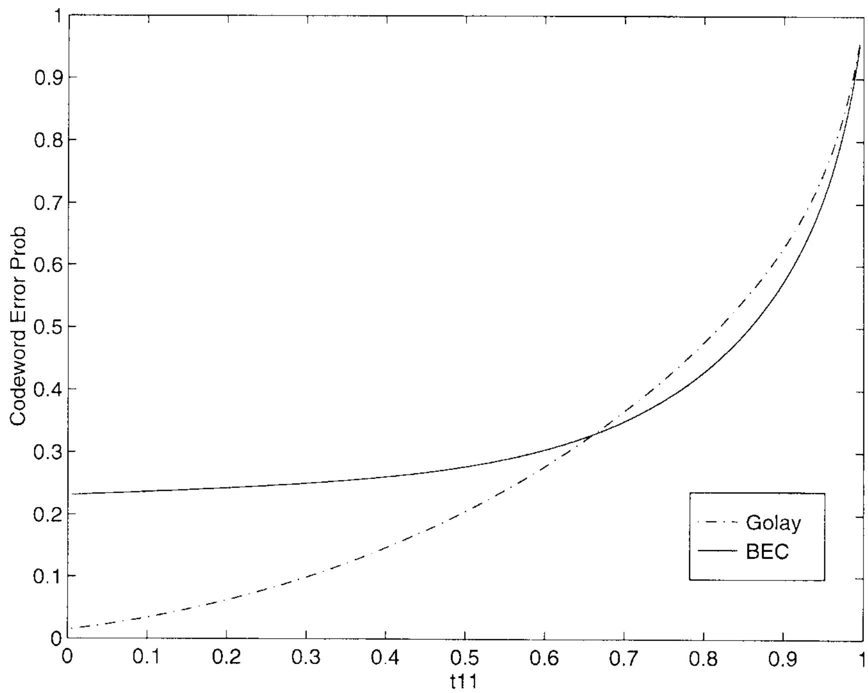

Fig. 2. Codeword-error probabilities versus t_{11} for the (23,12) Golay code and the (23,13) BEC code.

probability of codeword error on the GE channel is readily obtained as

$$P_w = \pi_0 P_{\text{ce}}(p_0) + \pi_1 P_{\text{ce}}(p_1) \tag{14}$$

where $P_{\text{ce}}(p)$ represents the probability of codeword error on a BSC with symbol-error probability p. For an REC code with block length N and RCC t

$$P_{\text{ce}}(p) \equiv P_{\text{ce}}^{\text{REC}}(N, t, p) = 1 - \sum_{i=0}^{t} \binom{N}{i} (1-p)^{N-i} p^i. \tag{15}$$

The corresponding expression for a cyclic BEC code with block length N, capable of correcting all cyclic bursts with cyclic burst length (CBL) $\leq l$, is obtained from

$$\begin{aligned} P_{\text{ce}}(p) \equiv P_{\text{ce}}^{\text{CBEC}}(N, l, p) &= 1 - \mathcal{P}(\text{No error}) \\ &\quad - \sum_{i=1}^{l} \mathcal{P}(\text{CBL} < l \text{ and } i \text{ errors}). \end{aligned} \tag{16}$$

Since on a BSC all error patterns with i errors are equiprobable

$$\begin{aligned} &\mathcal{P}(\text{CBL} < l \mid i \text{ errors}) \\ &\quad = \frac{\#\text{ of possible } i\text{-error patterns with CBL} < l}{\#\text{ of possible } i\text{-error patterns}} \\ &\quad = \frac{N\binom{(l-1)}{(i-1)}}{\binom{N}{i}} \end{aligned} \tag{17}$$

from which, using Bayes' rule, it follows that

$$\begin{aligned} P_{\text{ce}}^{\text{CBEC}}(N, l, p) &= 1 - (1-p)^N \\ &\quad - \sum_{i=1}^{l} N \binom{(l-1)}{(i-1)} (1-p)^{N-i} p^i. \end{aligned}$$

In a similar fashion, it can be shown that for a noncyclic BEC code with BCC of l

$$\begin{aligned} P_{\text{ce}}(p) &\equiv P_{\text{ce}}^{\text{BEC}}(N, l, p) \\ &= 1 - (1-p)^N - \sum_{i=1}^{l} (N + 1 - l) \binom{(l-1)}{(i-1)} \\ &\quad \cdot (1-p)^{N-i} p^i - \sum_{i=1}^{l} \sum_{r=1}^{l-i} \binom{(l-1-r)}{(i-1)} \\ &\quad \cdot (1-p)^{N-i} p^i. \end{aligned} \tag{18}$$

Note that the above expressions for probability of codeword error on a BSC are also useful under the assumption that the channel state changes very frequently, in which case the channel can itself be approximated by a BSC with the average bit-error rate $p_{\text{av}} = \pi_0 p_0 + \pi_1 p_1$, and the probability of codeword error is approximately given by $P_{\text{ce}}(p_{\text{av}})$.

V. Numerical Results

Consider the probability of codeword error on the GE channel for a pair of codes consisting of an REC code and a BEC code with similar redundancy and block length. One such pair is the (23,12) Golay code with an RCC of three and the (23,13) BEC code with a BCC of five obtained by shortening the (27,17) code in [11, p. 269]. Note that the Golay code is capable of correcting arbitrary error patterns provided the number of errors is less than or equal to three, which includes burst errors with burst length less than equal to three. Hence, for most values of the channel parameters, the Golay code has a lower probability of codeword error than the BEC code. However, if the GE channel produces a large number of error bursts with burst length between three and five, one can expect the BEC code to perform better than

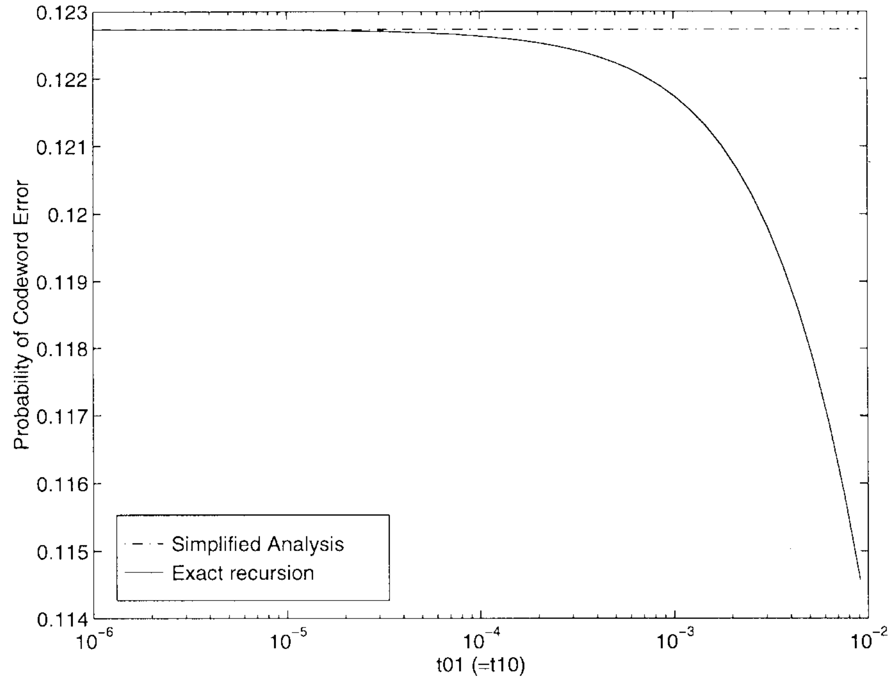

Fig. 3. Codeword-error probabilities versus t_{01} for the (23,13) BEC code obtained from exact recursions and simplified analysis.

the Golay code. One such scenario is demonstrated in Fig. 2, where the probability of codeword error for the two codes (obtained from the recursions of Section III) have been plotted against t_{11} (the probability for a state transition from the bad state back to the bad state), while the other channel parameters are held fixed at $t_{00} = 0.95$, $p_0 = 0.001$, $p_1 = 0.95$. From the figure, it is clear that at lower values of t_{11} the Golay code performs better than the BEC, but as the value of t_{11} is increased beyond 0.65 (approx.) the BEC code yields a lower probability of codeword error than the Golay code.

An example comparing the probability of codeword error for the BEC code obtained from the simplified analysis of Section IV with that from the exact recursion is shown in Fig. 3. For this example, the probability of bit error in the good and bad states were set to $p_0 = 0.001$ and $p_1 = 0.05$, respectively, and the state transition probabilities t_{01}, t_{10} were set equal to each other and varied over the range 10^{-6}–10^{-3}. From the graph (and also from direct reasoning), it can be seen that for this example the simplified analysis predicts no change in the probability of codeword error as we change $t_{01} = t_{10}$. For very small values of t_{10} and t_{01}, the approximation in the simplified analysis is valid and the results agree with the exact recursion. However, as the probability of state changes increases, the approximation breaks down and the results of the simplified analysis no longer agree with the exact recursions.

VI. Conclusion

This letter presents a modified analysis of the GE channel, which is useful for the performance evaluation of single BEC codes on these channels. Using recursions, expressions for the probability of codeword error for single BEC codes are obtained. For GE channels with extremely infrequent state changes, an alternate simplified analysis is also presented and shown to agree with the exact recursions.

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