Set Theoretic Signal Restoration Using an Error in Variables Criterion

Gaurav Sharma and H. Joel Trussell

Abstract— **In this correspondence, the restoration of a signal degraded by a stochastic impulse response is formulated as a problem with uncertainties in both the measurements and the impulse response. The method of total least squares, and variants thereof, are effective techniques for solving this class of problems. However, unlike set theoretic estimation schemes, these methods do not allow the incorporation of other *a priori* information in the estimate. In this correspondence, two new sets motivated by total least squares are introduced for set theoretic estimation. The convexity of these sets is established and the projection operators onto these sets are given. Through simulations, the advantages of the new technique over conventional and older set theoretic schemes for restoration are demonstrated.**

I. Introduction

Most signal restoration schemes assume that the impulse response of the degradation is known precisely. However, there are several physical situations where only the statistics of the degrading impulse response are available for use in restoration. In such cases, utilization of these statistics offers significant gains over processing based on averages alone.

For signals blurred by a linear system, the recorded data is given by

$$\mathbf{g}_{N\times 1} = \mathbf{H}_{N\times N}\mathbf{f}_{N\times 1} + \boldsymbol{\upsilon}_{N\times 1} \tag{1}$$

where the matrix $\mathbf{H}$ is the linear blur, $\mathbf{f}$ is the original signal to be restored, $\boldsymbol{\upsilon}$ is the measurement noise, and the subscripts indicate the dimensions of the vectors and matrices. For the case of stochastic blurs, $\mathbf{H}$ can be modeled as $\mathbf{H} = \bar{\mathbf{H}} + \boldsymbol{\Delta}\mathbf{H}$, where $\bar{\mathbf{H}}$ and $\boldsymbol{\Delta}\mathbf{H}$ represent the known and unknown parts of $\mathbf{H}$, respectively.

The method of total least squares (TLS), or error in variables regression, has been shown to be an effective technique for solving the set of equations in (1), where both $\mathbf{H}$ and $\mathbf{g}$ are contaminated with noise [1]–[3]. For shift-invariant blurs, the special structure of $\bar{\mathbf{H}}$ and $\boldsymbol{\Delta}\mathbf{H}$ can be exploited in the constrained TLS (CTLS) technique [4] to obtain better estimates. Recently, a regularized version of CTLS was demonstrated in [5], which attempts to preserve smoothness properties of the signal through the introduction of a regularization operator. While these estimation schemes are statistically sound, they do not permit use of other *a priori* knowledge of the signal in the estimation procedure.

Set theoretic estimation [6] provides a flexible framework for incorporation of *a priori* knowledge into estimates of the signal. If available knowledge about the signal can be represented in terms of sets $\{S_i\}_{i=1}^m$, in which the signal must lie, a point in the intersection of all these constraint sets, i.e., in

$$S_0 = \bigcap_{i=1}^{m} S_i$$

is used as an estimate for the signal. Obviously, set theoretic estimation is useful only for problems for which there exists a procedure for computing this estimate. If all the sets, $\{S_i\}_{i=1}^m$, are closed and convex, the method of successive projections onto convex sets (POCS) is guaranteed to converge to a point in S_0 starting from any arbitrary initial estimate [7], [8]. This result has been successfully exploited for a number of signal restoration problems with a known blur, and several convex sets have been defined based on prior knowledge and noise statistics for that case [9], [10]. In [11], the sets based on noise properties were modified to take into account the case of a stochastic blur. The modifications in [11] were primarily enlargements of the sets based on noise statistics, to incorporate the additional uncertainties introduced by inaccurate knowledge of the blur. While this simple approach is legitimate, it makes a rather limited use of the blur statistics. In this correspondence, motivated by the TLS approach, new sets describing the properties of the noise and the blur perturbations are defined in spatial and frequency domains. The convexity of the new sets is established by means of an elegant alternate characterization of these sets, and the projection operators onto these sets are developed based on the alternate characterization. Finally, through one-dimensional (1-D) and two-dimensional (2-D) simulations, the performance of restoration based on the modified sets is compared with the stochastic minimum mean-squared error (MMSE) filter and the approach in [11].

II. Sets Based on an Error in Variables Criterion

In this section, modifications of the noise variance and the power spectral bounds sets are considered to account for the stochastic variations in the blur. Throughout this discussion, $\mathbf{f}, \mathbf{g}, \boldsymbol{\upsilon}, \mathbf{x}, \boldsymbol{\eta} \in R^N$ and $\boldsymbol{\upsilon}$ will be assumed to be a zero mean white noise process with a variance of σ_v^2 and a Gaussian probability density function.

A. Modified Residual Variance Set

A set based on variance of the residual was defined, for the known blur case, in [10] as

$$S_v = \{\mathbf{x} \mid \|\mathbf{g} - \mathbf{H}\mathbf{x}\|^2 \le \delta_v\} \tag{2}$$

where δ_v was set equal to $N\sigma_v^2$. Since only $\bar{\mathbf{H}}$ is known for the case under consideration, the above set needs to be modified. In [11] the set was enlarged by considering the modified noise process $\boldsymbol{\upsilon}' = \mathbf{g} - \bar{\mathbf{H}}\mathbf{f} = \boldsymbol{\upsilon} + \boldsymbol{\Delta}\mathbf{H}\mathbf{f}$. The resulting set for sample variance of the residual was then obtained as

$$S_v' = \{\mathbf{x} \mid \|\mathbf{g} - \bar{\mathbf{H}}\mathbf{x}\|^2 \le \delta_v'\} \tag{3}$$

where $\delta_v' = \delta_v + E\|\boldsymbol{\Delta}\mathbf{H}\mathbf{f}\|^2$ is the variance of $\boldsymbol{\upsilon}'$, with E representing the expectation operator. For the purposes of restoration, the worst case value of $E\|\boldsymbol{\Delta}\mathbf{H}\mathbf{f}\|^2$ given by $E\|\boldsymbol{\Delta}\mathbf{H}\|^2\|\mathbf{f}\|^2$ was used.

An alternate approach can be used to account for the unknown component of $\mathbf{H}$. First, note that the set S_v can be rewritten as $S_v = \{\mathbf{x} \mid \exists \boldsymbol{\eta} \ni \mathbf{H}\mathbf{x} = \mathbf{g} + \boldsymbol{\eta}, \|\boldsymbol{\eta}\|^2 \le \delta_v\}$. Since least-squares

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G. Sharma is with the Digital Imaging Technology Center, Xerox Corporation, Webster, NY 14580 USA (e-mail: sharma@wrc.xerox.com).

H. J. Trussell is with the Electrical and Computer Engineering Department, North Carolina State University, Raleigh, NC 27695-7911 USA.

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restoration solves the optimization

$$\min_{\mathbf{x}} \|\boldsymbol{\eta}\|^2 \quad \text{subject to } \mathbf{Hx} = \mathbf{g} + \boldsymbol{\eta} \tag{4}$$

the method of least squares can be considered as the motivation for the definition of S_v. For a stochastic blur, the TLS method is statistically more appropriate than least squares [3]. The (weighted) TLS solution solves the optimization

$$\min_{\mathbf{x}} \tau\|\mathbf{E}\|_F^2 + \|\boldsymbol{\eta}\|^2 \quad \text{subject to } (\bar{\mathbf{H}} + \mathbf{E})\mathbf{x} = \mathbf{g} + \boldsymbol{\eta} \tag{5}$$

where $\|\cdot\|_F$ denotes the Frobenius norm [12] and τ is a positive weight determined by the statistics of $\boldsymbol{\Delta}\mathbf{H}$ and $\boldsymbol{v}$. Drawing on the analogy between least squares and S_v, a set can be defined based on TLS (or error in variables regression) as

$$\begin{aligned} S_{\text{TLS}} = \big\{\mathbf{x} \mid \exists\, \{\mathbf{E}, \boldsymbol{\eta}\} \ni (\bar{\mathbf{H}} + \mathbf{E})\mathbf{x} \\ = \mathbf{g} + \boldsymbol{\eta}, \tau\|\mathbf{E}\|_F^2 + \|\boldsymbol{\eta}\|^2 \le \nu\big\} \end{aligned} \tag{6}$$

where τ and ν are positive parameters determined by the statistics of $\boldsymbol{\Delta}\mathbf{H}$ and $\boldsymbol{v}$ (as will be described in Section IV).

In order to use the set S_{TLS} in POCS-based signal restoration, it is necessary to establish its convexity and to determine the projection onto it. The implicit definition in (6) involving $\mathbf{E}, \boldsymbol{\eta}$ makes both these tasks rather difficult. The following theorem (the proof of which appears in the Appendix) provides us an alternate characterization of S_{TLS} simplifying both tasks.

Theorem 1 [13]: Let $\Gamma = \{\mathbf{x} \in R^N \mid \|\bar{\mathbf{H}}\mathbf{x} - \mathbf{g}\|^2 - \frac{\nu}{\tau}\|\mathbf{x}\|^2 - \nu \le 0\}$, then $S_{\text{TLS}} = \Gamma$.

Corollary 1: S_{TLS} is closed and convex if $\sqrt{\frac{\nu}{\tau}}$ is less than or equal to the smallest singular value of $\mathbf{H}$.

Proof: Note that

$$\begin{aligned} f(\mathbf{x}) &\equiv \|\bar{\mathbf{H}}\mathbf{x} - \mathbf{g}\|^2 - \frac{\nu}{\tau}\|\mathbf{x}\|^2 - \nu \\ &= \mathbf{x}^T\Big(\bar{\mathbf{H}}^T\bar{\mathbf{H}} - \frac{\nu}{\tau}\mathbf{I}\Big)\mathbf{x} - 2\mathbf{g}^T\bar{\mathbf{H}}\mathbf{x} + \|\mathbf{g}\|^2 - \nu \end{aligned} \tag{7}$$

is a convex function of $\mathbf{x}$ if $\sqrt{\frac{\nu}{\tau}}$ is less than or equal to the smallest singular value of $\mathbf{H}$. Convexity of S_{TLS} follows immediately from the observation that $S_{\text{TLS}} = \{\mathbf{x} \mid f(\mathbf{x}) \le 0\}$. Closure follows from the continuity of $f(\mathbf{x})$. It is worth noting that the converse of the above corollary also holds provided $\|\mathbf{g}\|^2 > \nu$.

B. Modified Power Spectral Bounds (PSB) Set

If the blur is assumed to be shift invariant, the operators $\bar{\mathbf{H}}$ and $\boldsymbol{\Delta}\mathbf{H}$ are Toeplitz matrices determined by the known part, $\mathbf{h}$, and the unknown part, $\Delta\mathbf{h}$, of the stochastic impulse response, respectively. Under this assumption, (1) can be replaced by the following equivalent set of equations in the frequency domain:

$$\begin{aligned} G(k) &= (\bar{H}(k) + \Delta H(k))F(k) + \Upsilon(k) \\ &= H(k)F(k) + \Upsilon(k) \quad k = 0, 1, \cdots, (N-1) \end{aligned} \tag{8}$$

where upper case letters represent the discrete Fourier transform (DFT) of their lower case counterparts. Several sets analogous to the space domain sets can then be defined in the frequency-domain for use in set-theoretic restoration.

For the known blur case, convex sets were defined in [10] by placing appropriate confidence limits on $E\{|G(k) - H(k)F(k)|^2\}$, $0 \le k \le (N-1)$ (the periodogram of the residual). In [11], the sets were modified to account for stochastic impulse responses by expanding the bounds and making them frequency dependent, to get the sets

$$S_p^{(k)} = \{\mathbf{x} \mid |G(k) - \bar{H}(k)X(k)|^2 \le \delta'_p(k)\}, \quad 0 < k < \frac{N}{2} \tag{9}$$

where

$$\delta'_p(k) = \frac{\alpha N}{2}\sigma_v^2 + \frac{\alpha N}{2}P_f(k)E|\Delta H(k)|^2. \tag{10}$$

$P_f(k)$ is the periodogram of $\mathbf{f}$, and α is a confidence factor determined for a given confidence level for a normalized chi-squared random variable with two degrees of freedom.

The sets in (9) are frequency-domain equivalents of S'_v. Similar equivalents of S_{TLS} that are based on TLS can be defined as

$$\begin{aligned} S_t^{(k)} &= \{\mathbf{x} \mid \exists \Delta, E \in C \ni [\bar{H}(k) + \Delta]X(k) \\ &= G(k) + E, \tau_k|\Delta|^2 + |E|^2 \le \nu_k\}, \quad 0 < k < \frac{N}{2}. \end{aligned} \tag{11}$$

From arguments identical to those used in the proof of Theorem 1, it can be shown that

$$S_t^{(k)} = \left\{\mathbf{x} \mid |\bar{H}(k)X(k) - G(k)|^2 - \frac{\nu_k}{\tau_k}|X(k)|^2 - \nu_k^2 \le 0\right\}. \tag{12}$$

From the above characterization, it follows that $S_t^{(k)}$ is a closed convex set if $|\bar{H}(k)|^2 \ge \frac{\nu_k}{\tau_k}$.

III. Projections Onto the Modified Sets

Using the alternate characterizations developed above for the modified sets, projections can be readily determined using standard nonlinear programming techniques [14]. The projection of $\mathbf{y} \notin S_{\text{TLS}}$ onto S_{TLS} is given by

$$\mathbf{x}_0 = \left[\mathbf{I} + \lambda\Big(\bar{\mathbf{H}}^T\bar{\mathbf{H}} - \frac{\nu}{\tau}\mathbf{I}\Big)\right]^{-1}(\mathbf{y} + \lambda\bar{\mathbf{H}}^T\mathbf{g}) \tag{13}$$

where the Kuhn–Tucker parameter $\lambda \ge 0$ is determined so as to satisfy the constraint $f(\mathbf{x}_0) = 0$, where $f(\cdot)$ is as defined in (7). For shift-invariant blurs, the computation can be carried out efficiently by using the DFT to diagonalize $\bar{\mathbf{H}}$ [10].

The projection of $\mathbf{y} \notin S_t^{(k)}$ onto $S_t^{(k)}$ can be expressed in terms of the DFT as

$$X_0(l) = \begin{cases} \frac{1}{1+\lambda\left(|\bar{H}(k)|^2 - \frac{\nu_k}{\tau_k}\right)}(Y(k) + \lambda\bar{H}^*(k)G(k)), & l = k \\ Y(l), & l \ne k \end{cases} \tag{14}$$

where $\bar{H}^*(k)$ denotes the complex conjugate of $\bar{H}(k)$ and the Kuhn–Tucker parameter

$$\lambda = \frac{1}{\left(|\bar{H}(k)|^2 - \frac{\nu_k}{\tau_k}\right)} \times \left[1 + \sqrt{1 + \frac{|\bar{H}(k)X(k) - G(k)|^2 - \frac{\nu_k}{\tau_k}|X(k)|^2 - \nu_k^2}{\nu_k\left(1 + \frac{|G(k)|^2}{\left(|\bar{H}(k)|^2 - \frac{\nu_k}{\tau_k}\right)}\right)}}\right].$$

IV. Bounds for the Sets

The statistics of the noise $\boldsymbol{v}$ and the perturbations $\boldsymbol{\Delta}\mathbf{H}$ can be used to determine the values of the parameters $\nu, \tau, \{\nu_k, \tau_k\}_{k=1}^{N/2}$. For the modified noise variance set, the value of ν can be chosen as

$$\nu = \tau E\|\boldsymbol{\Delta}\mathbf{H}\|^2 + E\|\boldsymbol{v}\|^2$$

where τ is chosen large enough to ensure that the resulting set is convex (see Corollary 1), i.e.,

$$\tau \ge \frac{E\|\boldsymbol{v}\|^2}{\sigma_N^2(\bar{\mathbf{H}}) - E\|\boldsymbol{\Delta}\mathbf{H}\|^2}$$

where $\sigma_N^2(\bar{\mathbf{H}})$ denotes the smallest singular value of $\bar{\mathbf{H}}$. Note that the above condition for convexity of S_{TLS} is rather restrictive. In

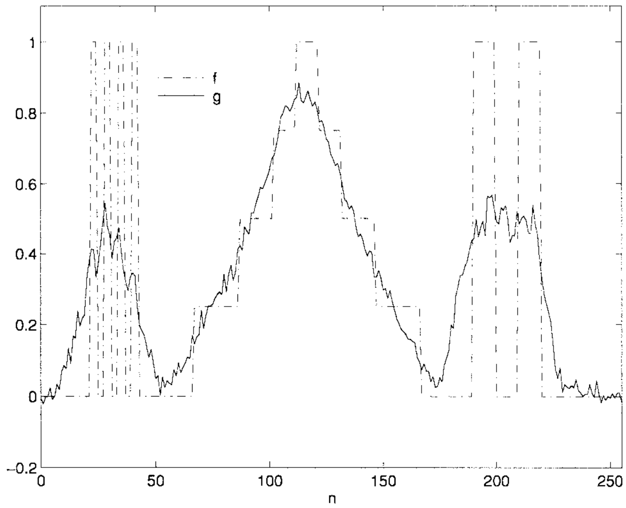

Fig. 1. Original and degraded signals.

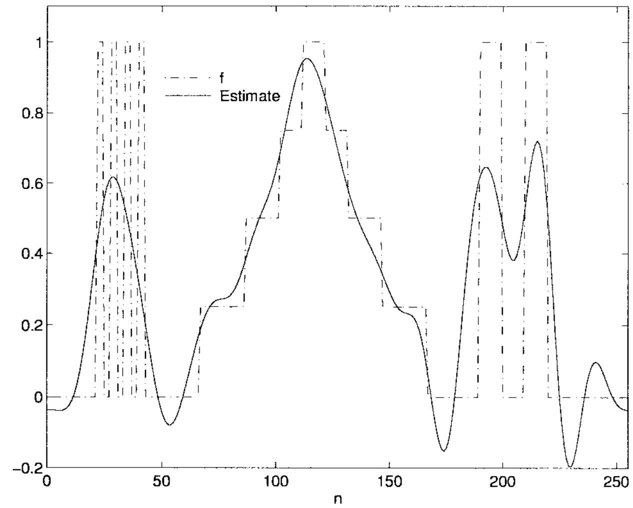

Fig. 2. Stochastic MMSE restoration.

particular, the above requirements implicitly assume that $\|\mathbf{\Delta H}\|^2 \leq \frac{\nu}{\tau} \leq \sigma_N^2(\bar{\mathbf{H}})$ holds for our physical model. For shift-invariant blurs, this translates to $\max|\Delta H(k)|^2 \leq \min|\bar{H}(k)|^2$. The set $S_{\rm TLS}$ is therefore useful only if the average blur $\bar{\mathbf{H}}$ is reasonably well conditioned in relation to its perturbations.

For the modified PSB set, the value of ν_k can be determined as

$$\begin{aligned}\nu_k &= \alpha\tau_k E|\Delta H(k)|^2 + \alpha E|\Upsilon(k)|^2 \\ &= \alpha\tau_k E|\Delta H(k)|^2 + \frac{\alpha N}{2}\sigma_v^2\end{aligned}$$

where α is a confidence factor for the required confidence level and τ_k is chosen large enough to ensure the convexity of $S_t^{(k)}$, i.e.,

$$\tau_k \geq \frac{\alpha N \sigma_v^2}{2(|\bar{H}(k)|^2 - \alpha E|\Delta H(k)|^2)}. \tag{15}$$

Once again, in deriving the above inequality, it is implicitly assumed that $|\bar{H}(k)|^2 \geq \alpha E|\Delta H(k)|^2$. Due to the lowpass nature of most blurs this requirement will be rarely met for the higher frequencies. Therefore, out of the sets $\{S_t^{(k)}\}_{k=1}^{N/2}$ only those may be retained for which $|\bar{H}(k)|^2 \geq \alpha E|\Delta H(k)|^2$. This is similar to [10] where the sets such that $|\bar{H}(k)| \approx 0$ were dropped.

V. Experimental Results

In order to test the effectiveness of the modified sets in restoration problems, 1-D and 2-D[1] simulations were performed using the stochastic blurs from [11]. In both cases, the stochastic blur was assumed to be composed of a known mean, $\bar{h}(n)$, and a random variation, $\Delta h(n)$, with known power spectrum $E\{|\Delta H(k)|^2\}$.

Restorations were performed using three different methods: the MMSE filter, the dynamic POCS (DPOCS) technique from [11], and POCS using the modified sets motivated by total least squares, referred to as the error in variables POCS (EVPOCS) method[2].

The stochastic MMSE filter (1-D version) is given by

$$Q(k) = \frac{P_f(k)\bar{H}^*(k)}{P_f(k)(|\bar{H}(k)|^2 + E|\Delta H(k)|^2) + \sigma_v^2}. \tag{16}$$

[1] For brevity, the mathematical expressions for the 1-D case only are used throughout this paper. The generalizations to 2-D signals are trivial.

[2] Since there is no minimization involved, the EVPOCS nomenclature is preferable to TLSPOCS even though the motivation for the new sets came from TLS.

For both the POCS schemes, the degraded signal was used as the initial estimate for starting the iterations. For the dynamic POCS technique of [11], the estimate was obtained by projecting sequentially onto the sets S_v', $\{S_p^{(k)}\}_{k=1}^{N/2}$, and S_n, the set of nonnegative vectors. The projections onto S_v' and $\{S_p^{(k)}\}$ are trivial modifications of those derived in [10] and will not be reproduced here. For the EVPOCS scheme, the set $S_{\rm TLS}$ was not used in the 1-D and 2-D simulations because in either case, the average impulse response is highly ill-conditioned, having actual zeros and near zero values in its DFT (see Section IV). The estimate for EVPOCS method was therefore found by projecting sequentially onto the sets S_v', $\{S_t^{(k)}\}_{k\in\Omega}$, and S_n, where the index set Ω was composed of all $1 \leq k \leq N/2$ for which the inequality $|\bar{H}(k)|^2 \geq \alpha E|\Delta H(k)|^2$ was true, where the confidence factor α was found by using a 99% confidence level for a chi-squared random variable with two degrees of freedom [11]. The values of $\{\tau_k\}_{k\in\Omega}$ were set as ten times their minimum values prescribed by (15). The restoration was not unduly sensitive to the values of τ_k provided a large enough value was chosen to ensure reasonable curvature for the quadratic inequality describing $S_t^{(k)}$.

A. 1-D Simulations

For generating a realization of the 1-D stochastic impulse response, M points were generated between -128 and 127, with the location of each point being a zero-mean Gaussian-distributed random variable (r.v.) with a standard deviation of eight. The stochastic impulse response was then computed as

$$h(n) = \frac{\text{number of points in } [n-0.5, n+0.5)}{M}. \tag{17}$$

The mean of the stochastic impulse response

$$\bar{h}(n) \approx \frac{1}{8\sqrt{2\pi}}\exp\left(-\frac{n^2}{128}\right) \tag{18}$$

was assumed to be the known part of the impulse response. $E\{|\Delta H(k)|^2\}$ was also assumed to be known and computed by averaging over 1000 realizations of the stochastic impulse response. The noise was generated using a Gaussian random number generator, and the variance σ_v^2 was computed from the signal-to-noise ratio (SNR), defined as $10\log_{10}(\|f\|^2/\sigma_v^2)$. The SNR was set at 30 dB for the simulations. The parameter M which governs the uncertainty of the impulse response was set equal to 100 as in [11].

The original signal and the degraded signal are shown in Fig. 1. The three estimates are shown in Figs. 2, 3, and 4 for the stochastic

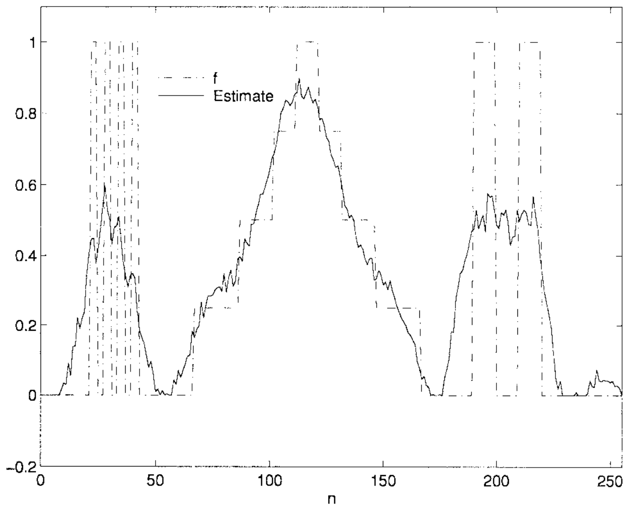

Fig. 3. Dynamic POCS restoration.

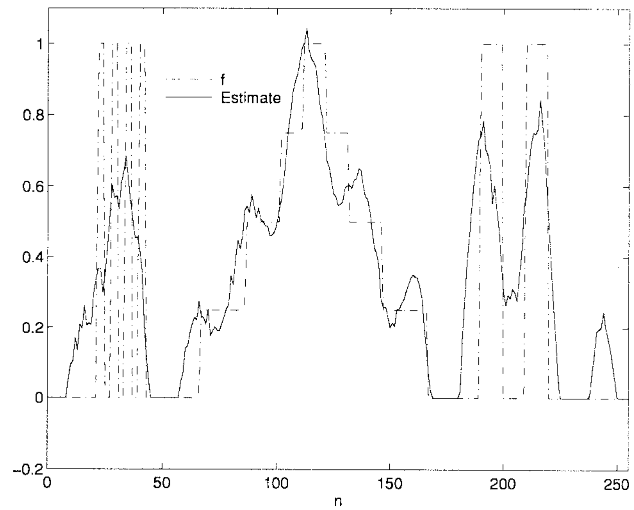

Fig. 4. Error in variables POCS estimate.

MMSE, DPOCS, and EVPOCS schemes, respectively. All plots include the actual signal for comparison. The DPOCS and EVPOCS restoration methods converged in five and eight iterations, respectively. From the figures, it can be seen that the stochastic MMSE filter yields a smooth restored signal that does not fully capture the levels in the original signal and has negative lobes. For the DPOCS technique, the sets are rather conservative and hence the estimate is not significantly different from the degraded signal used as the initial estimate. The EVPOCS approach using the sets $S_t^{(k)}$ performs significantly better giving good resolution of the two signal peaks around $n = 200$ and also some discrimination between the multiple signal levels in the central region.

To obtain mean-squared estimation error (MSEE) statistics for the different restorations, 1000 Monte Carlo simulations were performed with independent realizations of the stochastic blur and the noise. The simulation results were used to estimate the MSEE (in dB), defined as $10\log_{10}(E\{\|\mathbf{f}-\hat{\mathbf{f}}\|^2\}/\|\mathbf{f}\|^2)$, where $\mathbf{f}$ and $\hat{\mathbf{f}}$ are the the original and the restored signal, respectively. The DPOCS scheme failed to converge in 50 POCS iterations for 44 of the Monte Carlo runs, which were excluded from the corresponding MSEE computation. The MSEE for the stochastic MMSE, EVPOCS, and DPOCS (over converged iterations) restorations was estimated to be -6.34, -5.98, and -5.32 dB, respectively. Though the EVPOCS estimate has a slightly larger MSEE than the stochastic MMSE filter, which is expected, in almost every case the quantization steps were readily apparent and the double peaks on the right were better defined for EVPOCS. Note also that the stochastic MMSE filter requires knowledge of the signal periodogram, $\{P_f(k)\}_{k=0}^{N-1}$, which is not used in the POCS schemes. For the simulation example presented here, the known signal periodogram was utilized. In practice, the periodogram would have to be estimated, which would have a significant impact on the accuracy [15]. Since the degrading impulse response is stochastic in nature, the difficulty of the problem and the results of all schemes vary in accordance with the deviation of the stochastic blur realization from the mean. However, for typical realizations of the impulse response, the nature of the results is very similar to the example presented. Also, tests performed by using estimates of the noise variance that deviated from the true value by 10% indicate that all three restoration schemes are not overly sensitive to such errors.

Several additional points are worth noting with regard to this restoration problem. Note that the restoration of Fig. 4 has an apparent phase shift with respect to the original signal. This is to be expected, since the the known component of the blur is zero phase while the actual blur is not. For both the POCS restoration schemes, there is considerable latitude available in the estimate. For example, in either case, the stochastic MMSE estimate can be used as the initial estimate to get a much smoother restoration. Note also that the new sets can be readily incorporated in a POCS scheme that uses additional convex sets based on *a priori* and/or statistical information. Examples of such sets include, "smoothness" sets that define bounds on derivative norms [16]–[18] and noise outlier sets [10]. In particular, [17] demonstrates how smoothness may be enforced using POCS, while leaving edges intact.

B. 2-D Simulations

A 2-D stochastic point spread function (psf) $h(m,n)$ was obtained as the extension of the 1-D blur in (17). M points were generated in $R \times R$ such that the two coordinates of each generated point are independent identically distributed r.v.'s with standard deviation of four pixels. The stochastic impulse response was then computed as

$$h(m,n) = \frac{\text{number of points in } [m-0.5, m+0.5) \times [n-0.5, n+0.5)}{M}. \tag{19}$$

The average psf is therefore the zero-mean circularly symmetric 2-D Gaussian distribution with a variance of 16. As mentioned in [11] (where an image of the stochastic blur also appears), this blur resembles examples of short-exposure atmospheric psf and the psf resulting from photon scattering in X-ray imaging. As in the 1-D case, the known blur statistics are the average psf and $E\{|\Delta H(k)|^2\}$, the latter being computed by averaging over 1000 realizations of the stochastic impulse response. The parameter M that governs the uncertainty of the impulse response was set equal to 1000 as in [11].

For the simulations, the original 128×128 binary image of Fig. 5(a) was degraded by blurring it with a realization of the stochastic psf and adding stationary, zero-mean, Gaussian white noise, with variance corresponding to an SNR of 30 dB. The resulting degraded image is shown in Fig. 5(b). Restorations of the degraded image performed with the stochastic MMSE filter, DPOCS, and EVPOCS methods are shown in Fig. 5(c), (d) and (e), respectively. The presentation of results differs from the 1-D case in that (significantly large) negative lobes have been removed from the

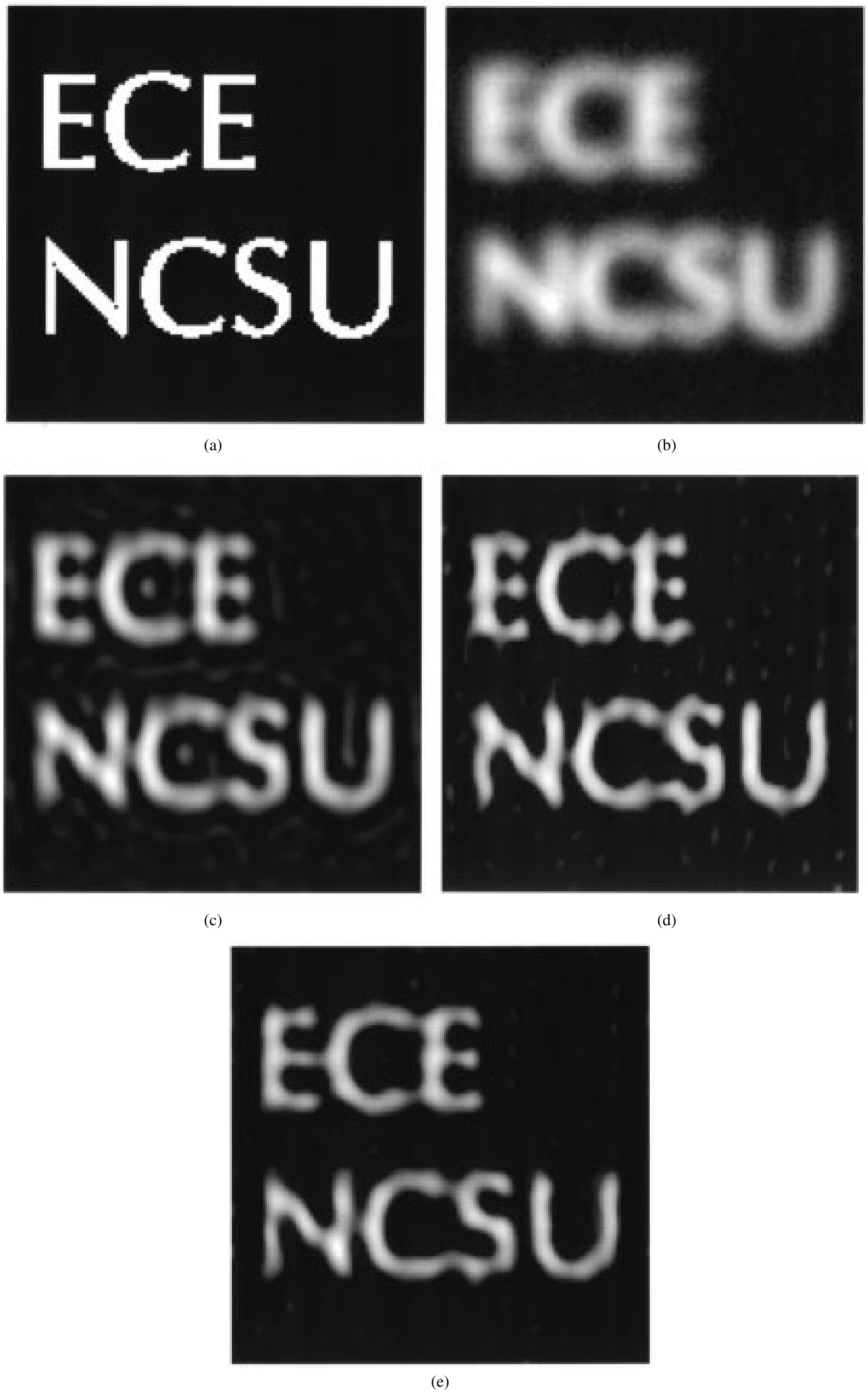

Fig. 5. Stochastic blur restoration example, $M = 100$. (a) Original image. (b) Degraded image. (c) Stochastic MMSE restoration. (d) DPOCS restoration. (e) EVPOCS restoration.

stochastic MMSE filter estimates for the purpose of image display. Apart from this, the characteristics of the restorations are similar to the 1-D case. The DPOCS and EVPOCS schemes yield sharper images and fewer reconstruction artifacts (in the image background) in comparison with the stochastic MMSE filter. In addition, both these schemes also give improved sharpness as compared to the stochastic MMSE filter. From the restorations, either of the POCS schemes cannot be seen to be uniformly better than the other. The letters in the EVPOCS restoration are better defined and it also has fewer artifacts in comparison with the DPOCS restoration but the spacing between the individual letters is clearer in the DPOCS restoration. However, as in the 1-D case, the EVPOCS scheme was seen to be more consistent with the data, i.e., for a given confidence level the EVPOCS method failed to converge less frequently than the DPOCS scheme.

VI. Conclusion

In this correspondence, new sets motivated by an error in variables criterion were proposed for use in set theoretic restoration of a signal blurred by a stochastic impulse response. An alternate characterization was developed for these sets, which helped establish their convexity and determine projections onto them for use in POCS based restoration. The superiority of the new technique over stochastic MMSE restoration and the dynamic POCS technique of [11], was demonstrated by means of simulations. The new approach is valuable for restoration of blurs that can be represented by stochastic models from which the required statistics can be determined. Examples of such blurs include those in x-ray imaging (due to the quantum nature of radiation) and in ground based astronomy (due to atmospheric perturbations) [19].

Appendix
Proof of Theorem 1

Suppose $\mathbf{x} \in S_{\mathrm{TLS}}$. Then there exists an $\mathbf{E}$ such that, $\tau\|\mathbf{E}\|_F^2 + \|\bar{\mathbf{H}}\mathbf{x} - \mathbf{g} + \mathbf{E}\mathbf{x}\|^2 - \nu \le 0$. Let

$$\beta = \begin{cases} \frac{\|\mathbf{E}\mathbf{x}\|}{\|\mathbf{x}\|} & \mathbf{x} \neq 0 \\ 0 & \mathbf{x} = 0 \end{cases}. \tag{20}$$

Then with a little arithmetic it can be seen that the above inequality involving $\mathbf{E}$ implies

$$(\tau + \|\mathbf{x}\|^2)\beta^2 - 2\|\bar{\mathbf{H}}\mathbf{x} - \mathbf{g}\|\|\mathbf{x}\|\beta + \|\bar{\mathbf{H}}\mathbf{x} - \mathbf{g}\|^2 - \nu \le 0. \tag{21}$$

Thus, if $\mathbf{x} \in S_{\mathrm{TLS}}$, the quadratic inequality (21) in β has a (nonnegative) real solution. This is equivalent to the discriminant being nonnegative, which simplifies to the condition $\|\bar{\mathbf{H}}\mathbf{x} - \mathbf{g}\|^2 - \frac{\nu}{\tau}\|\mathbf{x}\|^2 - \nu \le 0$. Hence, $\mathbf{x} \in \Gamma$.

Conversely, suppose $\mathbf{x} \in \Gamma$. Then from the discriminant rule used above it can be seen that there exists $\beta \ge 0$ satisfying inequality (21). Let β_0 be the minimum nonnegative solution of inequality (21). Let

$$\mathbf{E} = \begin{cases} \frac{-\beta_0}{\|\bar{\mathbf{H}}\mathbf{x} - \mathbf{g}\|\|\mathbf{x}\|}(\bar{\mathbf{H}}\mathbf{x} - \mathbf{g})\mathbf{x}^T & \beta_0 \neq 0 \\ 0 & \beta_0 = 0 \end{cases}. \tag{22}$$

Then it can be readily seen that

$$\begin{aligned} 0 &\ge \tau\beta_0^2 + \|\bar{\mathbf{H}}\mathbf{x} - \mathbf{g}\|^2 + \beta_0^2\|\mathbf{x}\|^2 - 2\beta_0\|\mathbf{x}\|\|\bar{\mathbf{H}}\mathbf{x} - \mathbf{g}\| - \nu \\ &= \tau\|\mathbf{E}\|_F^2 + \|\bar{\mathbf{H}}\mathbf{x} - \mathbf{g} + \mathbf{E}\mathbf{x}\|^2 - \nu. \end{aligned} \tag{23}$$

Hence, $\mathbf{x} \in S_{\mathrm{TLS}}$ and the theorem stands proved. Note that the theorem and the proof hold even if the Frobenius norm in the definition of S_{TLS} is replaced by the spectral norm.

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