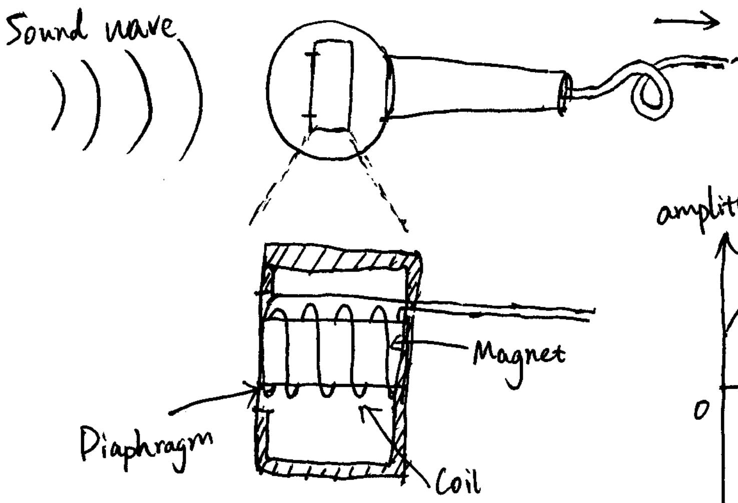
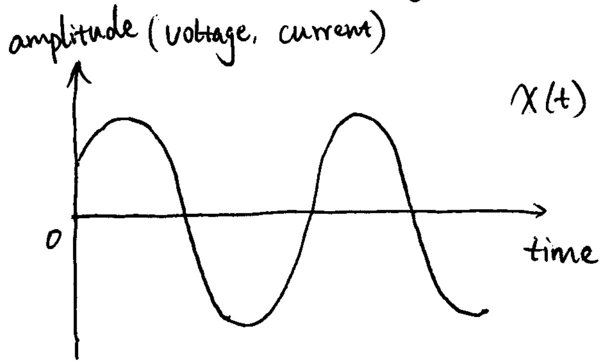
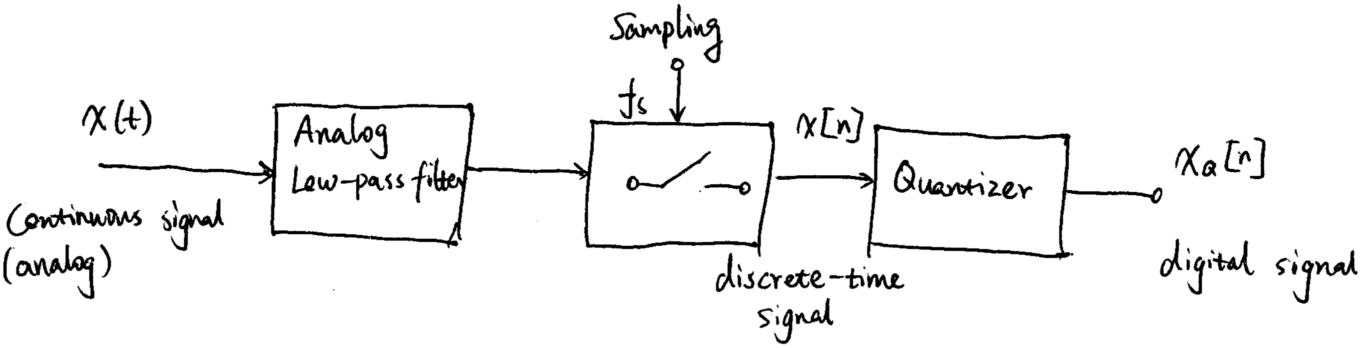
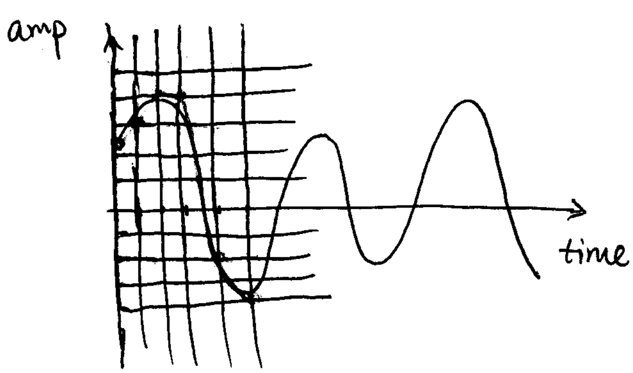
ECE 272/472: Lecture 1. Quantization



continuous audio signal.







Finer grids (in both time and amplitude)

$\Rightarrow$ better approximation.

Question: How fine should the grids be?

— 1) How frequent to sample the signal, i.e, sampling rate? (Nyquist-Shannon Sampling Theorem)

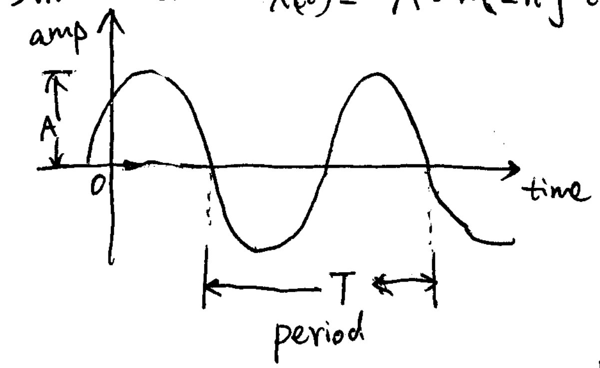
— 2) How many quantization levels? (Widrow Quantization Theorem).

Energy measure: Root-Mean-Square (RMS)

$x(t)$: $$x_{RMS} = \sqrt{\frac{1}{T_D}\int_{0}^{T_D} x^2(t)\,dt}$$ T_D: the length of the signal (or a long-enough chunk).

$x[n]$: $$x_{RMS} = \sqrt{\frac{1}{N}\sum_{n=0}^{N-1} x^2[n]}$$ N: length of the signal.

Sine Wave: $x(t) = A\sin(2\pi f t + \phi)$



frequency: $f = \frac{1}{T}$

angular freq: $\omega = 2\pi f = \frac{2\pi}{T}$

initial phase: ϕ

RMS of sine wave:

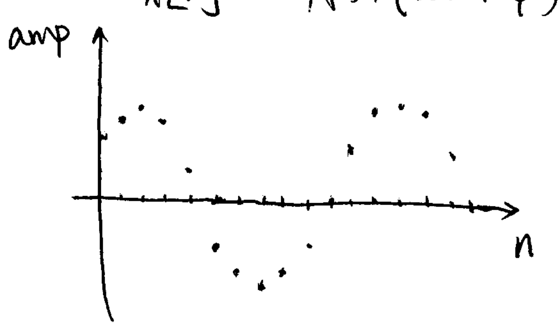
$$x_{RMS} = \sqrt{\frac{1}{T}\int_0^T A^2 \sin^2(2\pi f t + \phi)\,dt} \qquad (\cos 2x = 1 - 2\sin^2 x)$$

$$= \sqrt{\frac{1}{T}\int_0^T A^2\, \frac{1-\cos(4\pi f t + 2\phi)}{2}\,dt}$$

$$= \sqrt{A^2\, \frac{1}{T}\int_0^T \left(\frac{1}{2} - \frac{\cos(4\pi f t + 2\phi)}{2}\right)dt}$$

$$= \sqrt{A^2/2} \approx 0.707\,A$$

$x[n] = A\sin(\omega n + \phi)$



period: $N = \frac{2\pi}{\omega}$

Calculate RMS similarly.

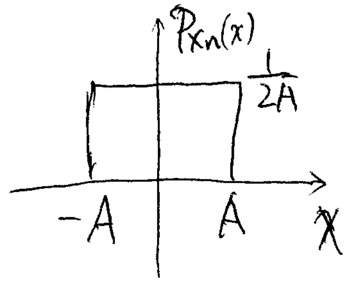
Model Signals as Random Processes

A signal $x[n]$ can be viewed as an instantiation (realization) of a sequence of random variables $\{X_n\}_{n=-\infty}^{\infty}$

Probability Density Function (PDF) of each random variable X_n (continuous valued)

$P_{X_n}(x)$, tells us the possible values that can be taken at sample n

e.g.



uniform distribution. $\frac{1}{2A}\,\mathrm{rect}\left(\frac{x}{2A}\right)$

$$P_{X_n}(x) = \begin{cases} \frac{1}{2A}, & \text{if } -A \le x \le A \\ 0, & \text{otherwise} \end{cases}$$

Moments of R.V.s

1st moment (mean): $\mathbb{E}X_n = \int_{-\infty}^{\infty} x\, P_{X_n}(x)\, dx \triangleq m_{X_n}$

2nd moment (averge power): $\mathbb{E}X_n^2 = \int_{-\infty}^{\infty} x^2\, P_{X_n}(x)\, dx$

Variance: $\mathrm{Var}(X_n) = \mathbb{E}(X_n - \mathbb{E}X_n)^2 \triangleq \delta_{X_n}^2$

when 0-mean: $\mathrm{Var}(X_n) = \mathbb{E}X_n^2$

m-th moment: $\mathbb{E}X_n^m = \int_{-\infty}^{\infty} x^m P_{X_n}(x)\, dx$

Joint PDF of two R.V.s: $P_{X_n, X_m}(x_n, x_m)$ describes their interdependency

X_n and X_m are <u>independent</u> iff $P_{X_n, X_m} = P_{X_n} \cdot P_{X_m}$

A random signal is <u>stationary</u> if all joint PDFs of any finitely many R.V.s are invariant of a shift of time, i.e.,

$$P_{X_n} = P_{X_{n+k}}$$

$$P_{X_n, X_m} = P_{X_{n+k}, X_{m+k}} \qquad \text{for all } k \in \mathbb{Z},\ l \in \mathbb{N}$$

$$P_{X_{t_1}, X_{t_2}, \cdots, X_{t_l}} = P_{X_{t_1+k}, X_{t_2+k}, \cdots, X_{t_l+k}}$$

For stationary random signals, we can calculate the statics by averaging.

mean $\hat{m}_X = \frac{1}{L} \sum_{n=0}^{L-1} x[n]$

Variance $\hat{\delta}_X^2 = \frac{1}{L} \sum_{n=0}^{L-1} (x[n] - \hat{m}_X)^2$

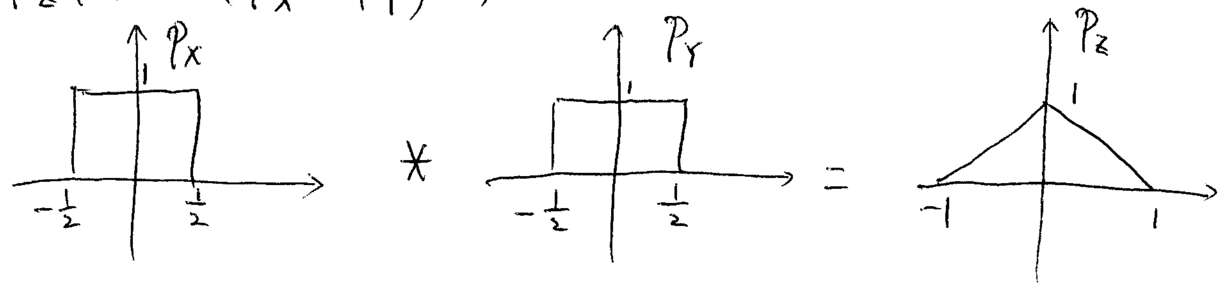
average power : $\frac{1}{L} \sum_{n=0}^{L-1} x^2[n] = X_{RMS}^2$

Sum of two independent R.V.s

Let X, Y, Z be R.Vs, $Z = X+Y$, X, Y independent.

$$P_Z(z) = (P_X * P_Y)(z)$$

e.g.



Characteristic Function (CF) of a R.V. is ~~it~~ the Fourier transform of its PDF.

$$P_X(x) \xrightarrow{\text{Fourier}} \mathcal{F}(P_X(x)) \triangleq \int_{-\infty}^{\infty} P_X(x) e^{-j2\pi u x} dx \triangleq F_X(u)$$

Now $\underbrace{F_Z(u)}_{\text{CF of } Z=X+Y} = \mathcal{F}(P_Z(z)) = \mathcal{F}((P_X * P_Y)(z)) = \mathcal{F}(P_X(z)) \cdot \mathcal{F}(P_Y(z))$

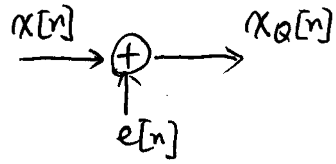
$$= \underbrace{F_X(u)}_{\text{CF of } X} \cdot \underbrace{F_Y(u)}_{\text{CF of } Y}$$

Quantization Error

$$x[n] \longrightarrow \boxed{\text{Quantizer}} \longrightarrow x_Q[n]$$

Error: $e[n] = x_Q[n] - x[n]$

Model error as an additive noise :



Quantization error clearly depends on quantization levels.

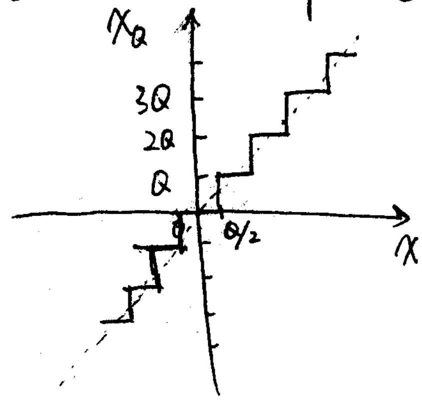
More levels $\Rightarrow$ smaller error

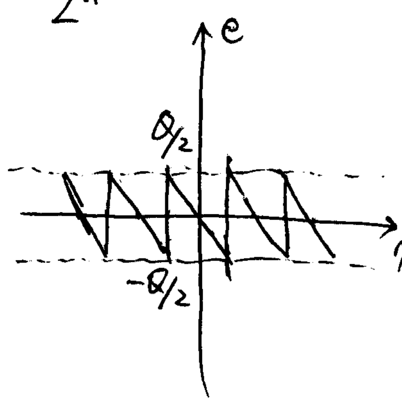
Encode levels with binary codes: N bits gives 2^N levels.

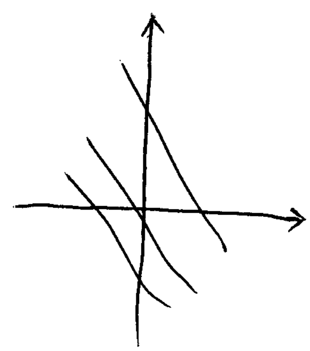
1	2
2	4
3	8
⋮	⋮

Suppose (without loss of generality) that the dynamic range of input signals is from -1 to $+1$. If we use N bits for quantization, then

Quantization Step: $Q = \dfrac{2}{2^N} = 2^{(1-N)}$



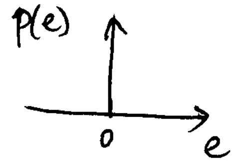




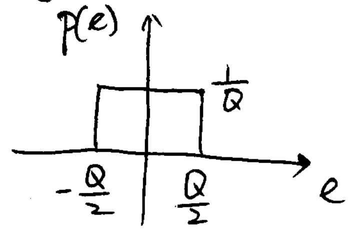
So Quantization Error is always in the range of $[-\frac{Q}{2}, \frac{Q}{2}]$

Q: If we view the error as a (stationary) random process, what is its PDF?
Will it depend on the input signal?

A: It depends on the input signal. E.g. if $x[n]$ happens to only have values at the quantization levels, then $e[n] \equiv 0$, i.e. $p(e)$ will be a Dirac function at $e=0$.



But in general, if the input signal has a wide dynamic range and the design of the quantizer is ~~d.~~ independent of the input, we can assume that $p(e)$ is uniform!



$$p(e) = \frac{1}{Q}\,\mathrm{rect}\left(\frac{e}{Q}\right)$$

So $m_e = 0$ (mean), $\sigma_e^2 = \int_{-\infty}^{\infty} e^2 p(e)\,de = \int_{-\infty}^{\infty} e^2 \frac{1}{Q}\mathrm{rect}\left(\frac{e}{Q}\right) de$

Variance (power, since 0-mean)

$$= \int_{-\frac{Q}{2}}^{\frac{Q}{2}} e^2 \cdot \frac{1}{Q}\, de = \frac{1}{Q}\cdot\frac{e^3}{3}\Big|_{-\frac{Q}{2}}^{\frac{Q}{2}} = \frac{1}{3Q}\left(\frac{Q^3}{8}+\frac{Q^3}{8}\right)$$

$$= \frac{Q^2}{12}$$

Suppose the input $x[n]$ is a sinusoid, spanning ~~of~~ the full dynamic range, with 0 mean, we know its $RMS = \sqrt{\frac{A^2}{2}}$, i.e. its ~~Variance~~ power (also σ_x^2) $= RMS^2 = \frac{A^2}{2} = \frac{1}{2}$.

Signal-to-Noise Ratio (SNR):

$$SNR = 10\log_{10}\left(\frac{\sigma_x^2}{\sigma_e^2}\right) \text{ dB}$$

$$= 10\log_{10}\left(\frac{\frac{1}{2}}{\frac{Q^2}{12}}\right) = 10\log_{10}\left(\frac{6}{2^{(1-N)\cdot 2}}\right)$$

$$= 10\log_{10}\left(2^{2N}\cdot\frac{6}{4}\right)$$

$$= N\cdot 20\log_{10}2 + 10\log_{10}\frac{3}{2}$$

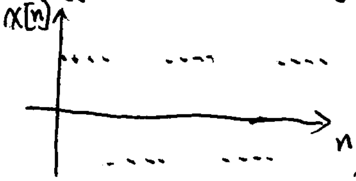
$$\approx 6.02N + 1.76 \quad \text{dB}$$

$\therefore$ if $N = 16$ (used in CD), SNR $\approx$ 98.08 dB for sine waves

Increase 1 bit (i.e., double the quantization level), will increase SNR for 6 dB.

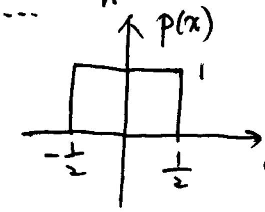
Apparently, for different input signals, SNR will be different.

for square wave



$\sigma_x^2 = 1$, $\sigma_e^2 = 0$. , $\text{SNR} \approx \cancel{101\,\text{dB}}$ ∞.

for uniformly distributed $x[n]$

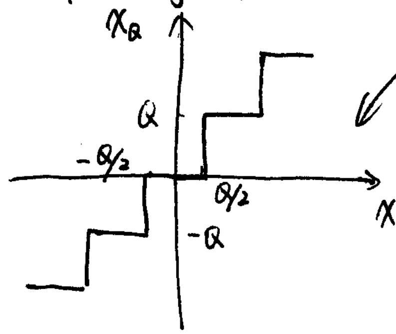


$\sigma_x^2 = \frac{1}{12}$, $\text{SNR} \approx 90\ \text{dB}$

for Gaussian distributed $x[n]$, if $\sigma_x^2 = \frac{1}{4}$, $\text{SNR} \approx 95\ \text{dB}$

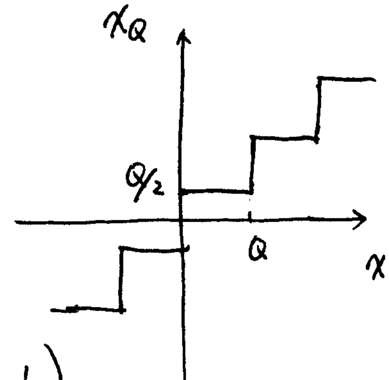
Statistical Analysis of Quantization Error

Two types of quantization:



midtread

$$x_Q[n] = Q\left\lfloor \frac{x[n]}{Q} + \frac{1}{2} \right\rfloor$$



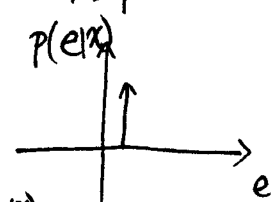
$$x_Q[n] = Q\left(\left\lfloor \frac{x[n]}{Q} \right\rfloor + \frac{1}{2}\right)$$

midriser

Their error analyses are similar. Let's assume midtread.

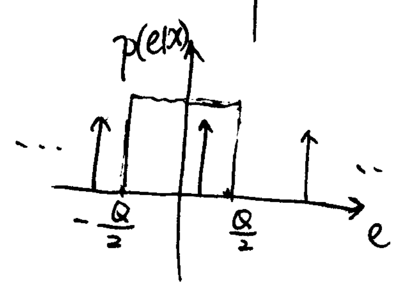
Given an input x, error is determined, i.e. conditional PDF

$$p_{E|X}(e|x) = \delta\left(e - (x_Q - x)\right) = \delta\left(e - Q\left\lfloor \frac{x}{Q} + \frac{1}{2} \right\rfloor + x\right)$$



$$= \text{Rect}\left(\frac{e}{Q}\right) \sum_{k=-\infty}^{\infty} \delta(e - kQ + x)$$

rectangular function $\times$ impulse train.



$$p_E(e) = \int_{-\infty}^{+\infty} p_{E|X}(e|x) \cdot p_X(x)\, dx$$

$$= \int_{-\infty}^{+\infty} \text{Rect}\left(\frac{e}{Q}\right) \sum_{k=-\infty}^{\infty} \underline{\delta(e - kQ + x) \cdot p_X(x)}\, dx$$

nonzero only when $x = -e + kQ$.

$$= \text{Rect}\left(\frac{e}{Q}\right) \sum_{k=-\infty}^{\infty} p_X(-e + kQ)$$