$$= \mathrm{Rect}\left(\frac{e}{Q}\right) \cdot \left[P_x(-e) * \delta_Q(e) \right]$$

where $\delta_Q(e) = \sum_{k=-\infty}^{\infty} \delta(e - kQ)$

In general: $$f(x) * \delta_Q(x) = \cancel{\Sigma} \int_{-\infty}^{\infty} f(\tau)\, \delta_Q(x-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} f(x-\tau)\, \delta_Q(\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} f(x-\tau) \sum_{k=-\infty}^{\infty} \delta(\tau - kQ)\, d\tau$$

$$= \sum_{k=-\infty}^{\infty} f(-kQ + x)$$

The characteristic function of the error

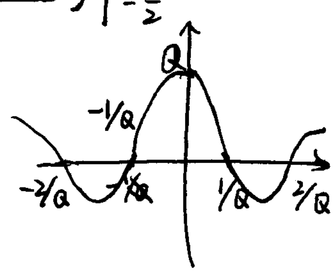
$$F_E(u) = \mathcal{F}(P_E(e)) = \mathcal{F}\left(\mathrm{Rect}\left(\frac{e}{Q}\right)\right) * \left[\mathcal{F}(P_x(-e)) \cdot \mathcal{F}(\delta_Q(-e)) \right]$$

$$\mathcal{F}\left(\mathrm{Rect}\left(\frac{e}{Q}\right)\right) = \int_{-\frac{Q}{2}}^{\frac{Q}{2}} 1 \cdot e^{-j2\pi u e}\, de \neq \cancel{\left.\frac{e^{-j2\pi u e}}{j2\pi u}\right|_{-\frac{Q}{2}}^{\frac{Q}{2}}} =$$

$$= \int_{-\frac{Q}{2}}^{\frac{Q}{2}} \left[\cos(2\pi u e) - j \sin(2\pi u e) \right] de$$

$$= \left. \frac{\left[\sin(2\pi u e) + j \cos(2\pi u e) \right]}{2\pi u} \right|_{-\frac{Q}{2}}^{\frac{Q}{2}}$$

$$= \frac{\sin(\pi u Q)}{\pi u}$$



$\mathcal{F}(P_x(-e)) = F_x(-u)$: CF of input X.

$$\mathcal{F}(\delta_Q(-e)) = \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \delta(-e - kQ)\, e^{-j2\pi u e}\, de$$

$$= \sum_{k=-\infty}^{\infty} e^{-j2\pi u(-kQ)} = \frac{1}{Q} \sum_{k=-\infty}^{\infty} \delta\left(u - \frac{k}{Q}\right)$$

Therefore, $$F_E(u) = \frac{\sin(\pi u Q)}{\pi u} * \left[F_X(-u)\cdot \frac{1}{Q}\sum_{k=-\infty}^{\infty} \delta\left(u-\frac{k}{Q}\right)\right]$$

$$= \frac{\sin(\pi u Q)}{\pi u} * \left[\frac{1}{Q}\sum_{k=-\infty}^{\infty} F_X\left(-\frac{k}{Q}\right)\delta\left(u-\frac{k}{Q}\right)\right]$$

$$= \sum_{k=-\infty}^{\infty} F_X\left(-\frac{k}{Q}\right)\frac{\sin\left(\pi Q\left(u-\frac{k}{Q}\right)\right)}{\pi Q\left(u-\frac{k}{Q}\right)}$$

Sripad & Snyder ~~Schuchman~~ Condition)

If $F_X(u)=0$ for $u=\frac{k}{Q}$ for all $k=\pm 1, \pm 2, \cdots$

then $F_E(u) = F_X(0)\cdot\frac{\sin(\pi Q u)}{\pi Q u}$

↑ CF of X when $u=0$. since $F_X(0)=1 = \int_{-\infty}^{+\infty} p(x)e^{-j2\pi 0\cdot x}\,dx = 1$

$\therefore P_E(e) = \frac{1}{Q}\,\mathrm{rect}\left(\frac{e}{Q}\right)$, i.e. uniform distribution.

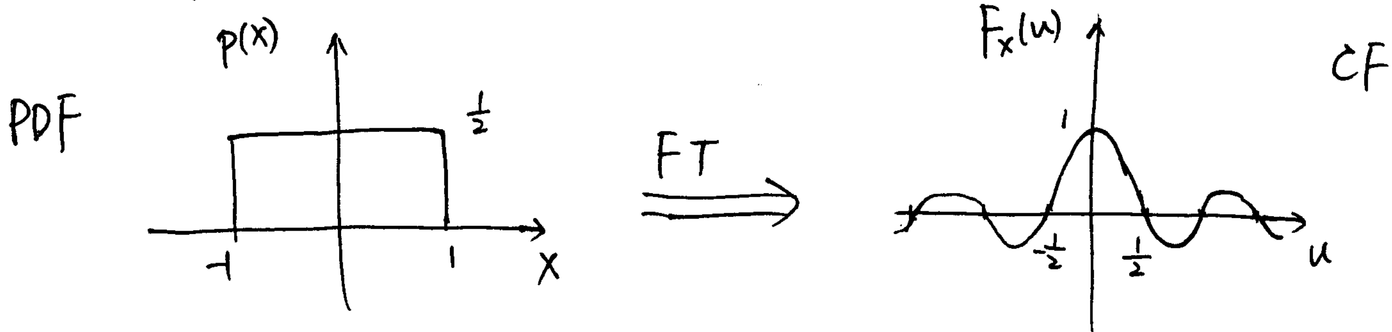
Widrow condition)

Or if $F_X(u)=0$, for all $|u| > \frac{1}{Q}$, there is also only one term in $F_X(u)$.

This is a much stronger condition

Since input $X[n]$ is always has a limited dynamic range, i.e. the support of its PDF is finite, ⇒ CF has infinite support, ⇒ Widrow condition is never satisfied.

But if X is uniformly distributed in $[-1, 1)$



As $Q = 2^{(1-N)}$, $\therefore \frac{k}{Q} = 2^{N-1}k$ is clearly a integer multiple of $\frac{1}{2}$.

∴ Schuchman Condition is ~~saf~~ satisfied.

Second-Order Statistics of error:

Important because it determines the power spectral characteristics of error signal.
We'll analyze joint PDF $p(e_n, e_m)$. First, look at conditional joint PDF given input x_n and x_m.

$$p(e_n, e_m \mid x_n, x_m) = \delta(e_n - (x_{Q_n} - x_n))\,\delta(e_m - (x_{Q_m} - x_m))$$

$$= \mathrm{Rect}\left(\frac{e_n}{Q}\right)\sum_{k=-\infty}^{\infty}\delta(e_n - kQ + x_n)\ \mathrm{Rect}\left(\frac{e_m}{Q}\right)\sum_{l=-\infty}^{\infty}\delta(e_m - lQ + x_m)$$

Similar to First-Order Statistical analysis,

$$p(e_n, e_m) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(e_n, e_m \mid x_n, x_m)\, p(x_n, x_m)\, dx_n\, dx_m$$

$$= \mathrm{Rect}\left(\frac{e_n}{Q}\right)\mathrm{Rect}\left(\frac{e_m}{Q}\right)\left[P_{X_n,X_m}(-e_n, -e_m) * \delta_{QQ}(-e_n, -e_m)\right]$$

where $\delta_{QQ}(e_n, e_m) = \sum_{k=-\infty}^{\infty}\delta(e_n - kQ)\sum_{l=-\infty}^{\infty}\delta(e_m - lQ)$

Take 2-D Fourier transform to get CF

$$F_{E_nE_m}(u_n, u_m) = \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} F_{X_n,X_m}\left(-\frac{k}{Q}, -\frac{l}{Q}\right)\frac{\sin\left(\pi Q\left(u_n - \frac{k}{Q}\right)\right)}{\pi Q\left(u_n - \frac{k}{Q}\right)}\cdot\frac{\sin\left(\pi Q\left(u_m - \frac{l}{Q}\right)\right)}{\pi Q\left(u_m - \frac{l}{Q}\right)}$$

(Sripad & Snyder Condition)

If $F_{X_n,X_m}(\cancel{u_n, u_m}\ {}^{k/Q,\ l/Q}) = 0$ for all integers $\cancel{u_n, u_m}\ {}^{k,\ l}$ with $\cancel{(u_n, u_m)}\ {}^{(k,l)} \neq (0,0)$

then
$$F_{E_nE_m}(u_n, u_m) = \underbrace{\frac{\sin\left(\pi Q\left(u_n \cancel{- \frac{k}{Q}}\right)\right)}{\pi Q\left(u_n \cancel{- \frac{k}{Q}}\right)}}_{\text{only about } u_n}\cdot\underbrace{\frac{\sin\left(\pi Q\left(u_m \cancel{- \frac{l}{Q}}\right)\right)}{\pi Q\left(u_m \cancel{- \frac{l}{Q}}\right)}}_{\text{only about } u_m}$$

$\therefore$ e_n and e_m are independent! Also they are both uniformly distributed! between $-\frac{Q}{2}$ and $\frac{Q}{2}$.

$\therefore$ All moments $\mathbb{E}(E_n^k E_m^l) = \mathbb{E}(E_n^k)\,\mathbb{E}(E_m^l)$

In particular, Autocorrelation
$$\mathbb{E}(E_n E_m) = \mathbb{E}(E_n)\,\mathbb{E}(E_m) = \begin{cases}\mathbb{E}(E_n^2) = \frac{Q^2}{12}, & \text{if } n = m\\ \mathbb{E}(E_n)\,\mathbb{E}(E_m) = 0, & \text{if } n \neq m\end{cases}$$

According to Wiener–Khinchin Theorem, Power Spectral Density (PSD) is the discrete-time Fourier transform (DTFT) of autocorrelation function.

So the PSD of error signal is flat, i.e., white!

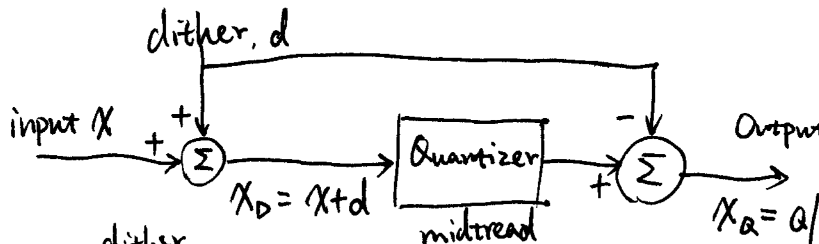
<u>Summary</u>: If the input signal satisfies <u>certain conditions</u>, the 1-st order statistical analysis tells us that the PDF of error is uniform, i.e. SNR is quantified; the 2-nd order statistical analysis tells us that the PSD of error is white, i.e. we will not hear artifacts.

Problem: The "certain conditions" are hard to satisfy.

Solution: Add "dither" to input signal before quantizing.

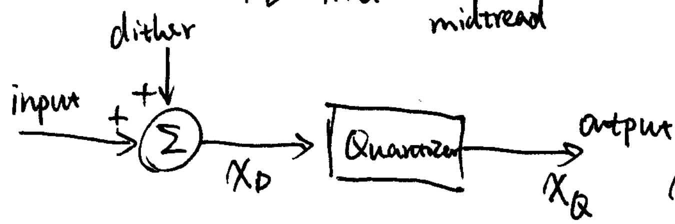
Dither signal is ~~an~~ ~~independent~~ randomly generated signal, independent to the input.

1) Subtractive dither



$$x_Q = Q\left\lfloor \frac{x[n]+d[n]}{Q} + \frac{1}{2} \right\rfloor - d$$

2) Nonsubtractive dither



$$x_Q = Q\left\lfloor \frac{x[n]+d[n]}{Q} + \frac{1}{2} \right\rfloor$$

1) gives best theoretic performance, but not convenient to implement

2) --- good-enough ------, and ~~ea~~ is easy -----.

Statistical Analysis of Quantization Error of Subtractive Dither

$$e = x_Q - x = Q\left\lfloor \frac{x+d}{Q} + \frac{1}{2} \right\rfloor - d - x = Q\left\lfloor \frac{x_D}{Q} + \frac{1}{2} \right\rfloor - x_D$$

So the analysis will be the same if we view x_D as input!

Therefore, to get nice properties of error, we now require x_D to satisfy those conditions. Since $x_D = x + d$ and x and d are independent, PDF of x_D will be the convolution of PDF of x and PDF of d, i.e. CF of x_D will be multiplication of CFs of x and d.

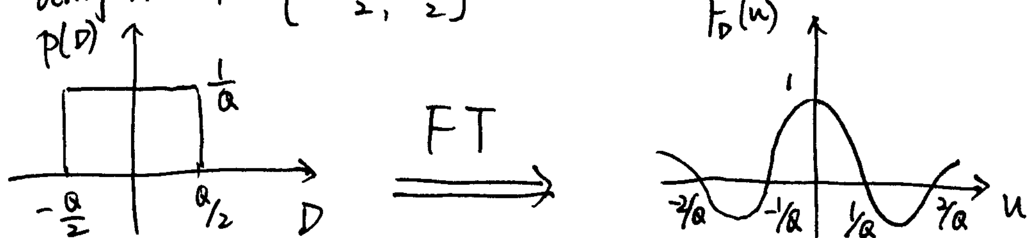
$$\therefore\ p_E(e) = \mathrm{Rect}\left(\frac{e}{Q}\right)\left[p_X(e) * p_D(-e) * \delta_Q(-e)\right]$$

$$\therefore\ F_E(u) = \frac{\sin(\pi u Q)}{\pi u} * \left[F_X(-u)\cdot F_D(-u)\cdot \frac{1}{Q}\sum_{k=-\infty}^{\infty}\delta\left(u-\frac{k}{Q}\right)\right]$$

(Schuchman's Condition)

If CF of dither $\cancel{F_D(u)=0}$ $F_D\left(\frac{k}{Q}\right)=0$ for all $k=\pm1,\pm2,\cdots$ then PDF of error is uniform in $\left[-\frac{Q}{2}, \frac{Q}{2}\right]$

This condition is easy to satisfy, as we can let the PDF of dither be uniform in $\left[-\frac{Q}{2}, \frac{Q}{2}\right]$



Or many other distributions, e.g. uniform in $[-Q, Q]$
triangular in $[-Q, Q]$

Similarly, for 2-nd order ~~condition~~ analysis, ~~we~~ if $F_{D_nD_m}\left(\frac{k}{Q}, \frac{l}{Q}\right)=0$ for all integers k and l with $(k,l)\neq(0,0)$, the PSD of error will be white

This is great as it means we can choose an appropriate dither signal to achieve nice properties of error, regardless of the input signal!
The error signal now becomes statistically independent of input.

For Nonsubtractive dither, we don't have this independence, but we can choose dither to make the moments of error independent of input.

Read the paper, especially Section 1, 2, 6, and Summaries of 3, 4, 5.
Figure 4, 6 and 10 are very insightful.