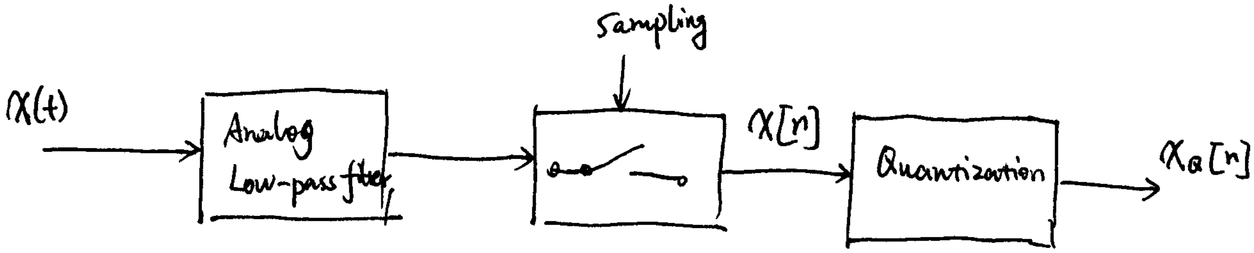
# ECE 272/472. Lecture 2: Sampling



Remember the basic A/D conversion diagram



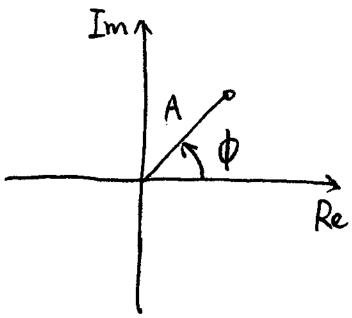

Questions: 1. Why do we need the Analog LP filter?

2. How fast should we sample the signal?

(Nyquist – Shannon Sampling Theorem)

Intuitive Explanation:

Consider $x(t)$ as a complex sinusoidal signal $x(t) = A e^{j(\omega t + \phi)}$



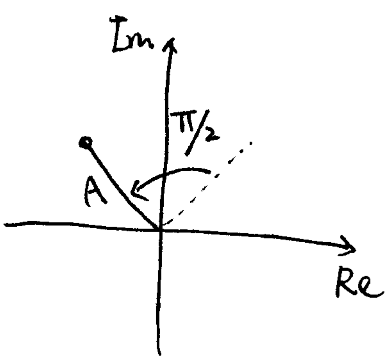

t = 0

$\omega$: angular frequency: how many radians per second

$f$: frequency: how many cycles per second

1 cycle = $2\pi$ radians

$\therefore f = \frac{\omega}{2\pi}$, $T = \frac{1}{f} = \frac{2\pi}{\omega}$

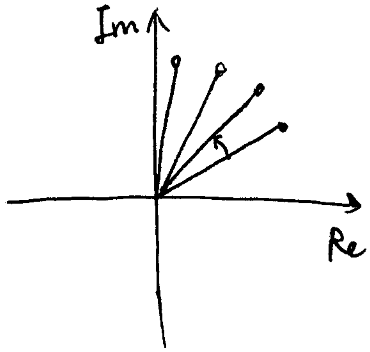



$\frac{\pi}{2}$ radians have passed

$\therefore t = \frac{\pi}{2} \cdot \frac{1}{\omega}$ in this figure.

Sampling: taking snapshots. $f_s$: Sampling rate/freq.: How many snapshots per second.

$T_s$: Sampling period: $T_s = \frac{1}{f_s}$




$$\frac{f}{f_s} = \frac{\frac{\#\text{ cycles}}{1\text{ sec}}}{\frac{\#\text{ samples}}{1\text{ sec}}} = \frac{\#\text{ cycles}}{1\text{ sample}}$$

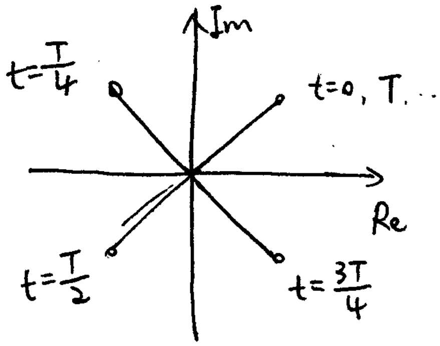
$$\therefore \frac{f}{f_s}\cdot 2\pi = \frac{\#\text{ radians}}{\text{Sample}} = \left(\text{phase advance between samples}\right).$$

$$\frac{f}{f_s}\cdot 2\pi = \frac{\omega}{f_s}$$

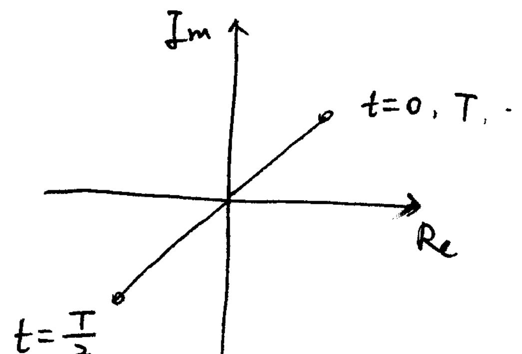
If $\frac{f}{f_s} = \frac{1}{10}$, then there are 10 samples/snapshots per cycle.

" " $\frac{f}{f_s} = \frac{1}{4}$, " " " " 4 " " " " " " ".

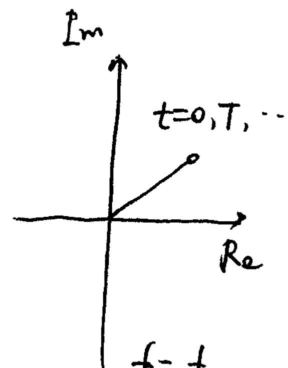
" " $\frac{f}{f_s} = \frac{1}{2}$, " " " " 2 " " " " " "




we see the motion pretty well.



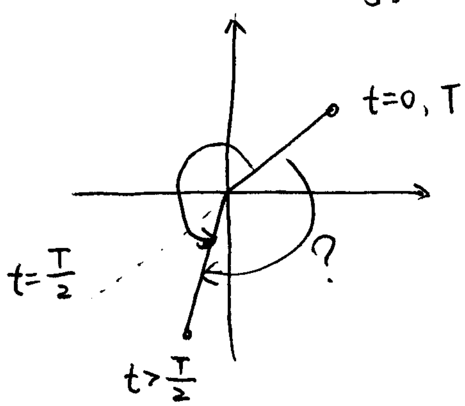

We can still see the motion




$f = f_s$

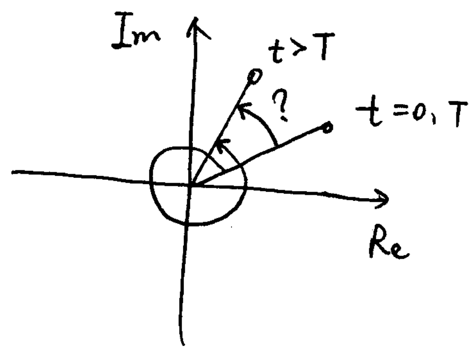
Seems not move

If $\frac{1}{2} < \frac{f}{f_s} < 1$




Seems like going backward with a smaller frequency.

If $\frac{f}{f_s} > 1$



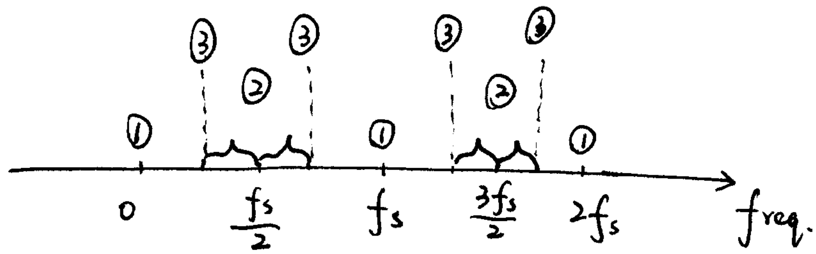

Seems like going forward with a much smaller frequency

To summerize: If sampling rate is $f_s$

Sinusoidal signals with $f = 0,\ f_s,\ 2f_s,\ \cdots$ are ambiguous to each other

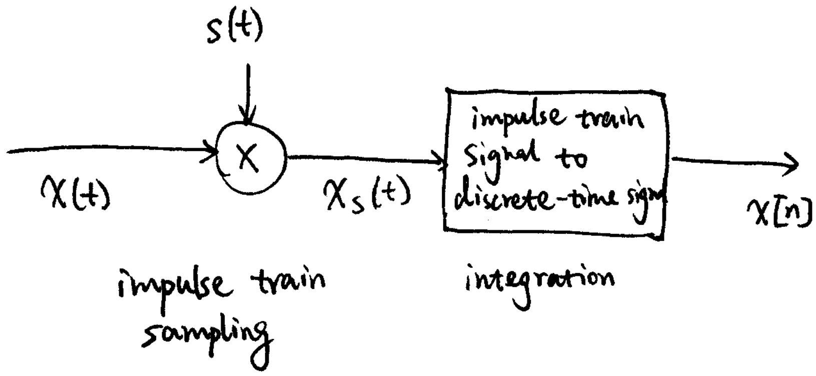
$f = \frac{f_s}{2},\ \frac{3f_s}{2},\ (k+\frac{1}{2})f_s,$ are ......

if $0 < f < \frac{f_s}{2}$, it's ambiguous to $f_s - f$.




Mathematics: $x[n] = x(nT_s)$

It's more convenient to represent this process in two stages




$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

↑ Dirac function

$$x_s(t) = x(t) \cdot s(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

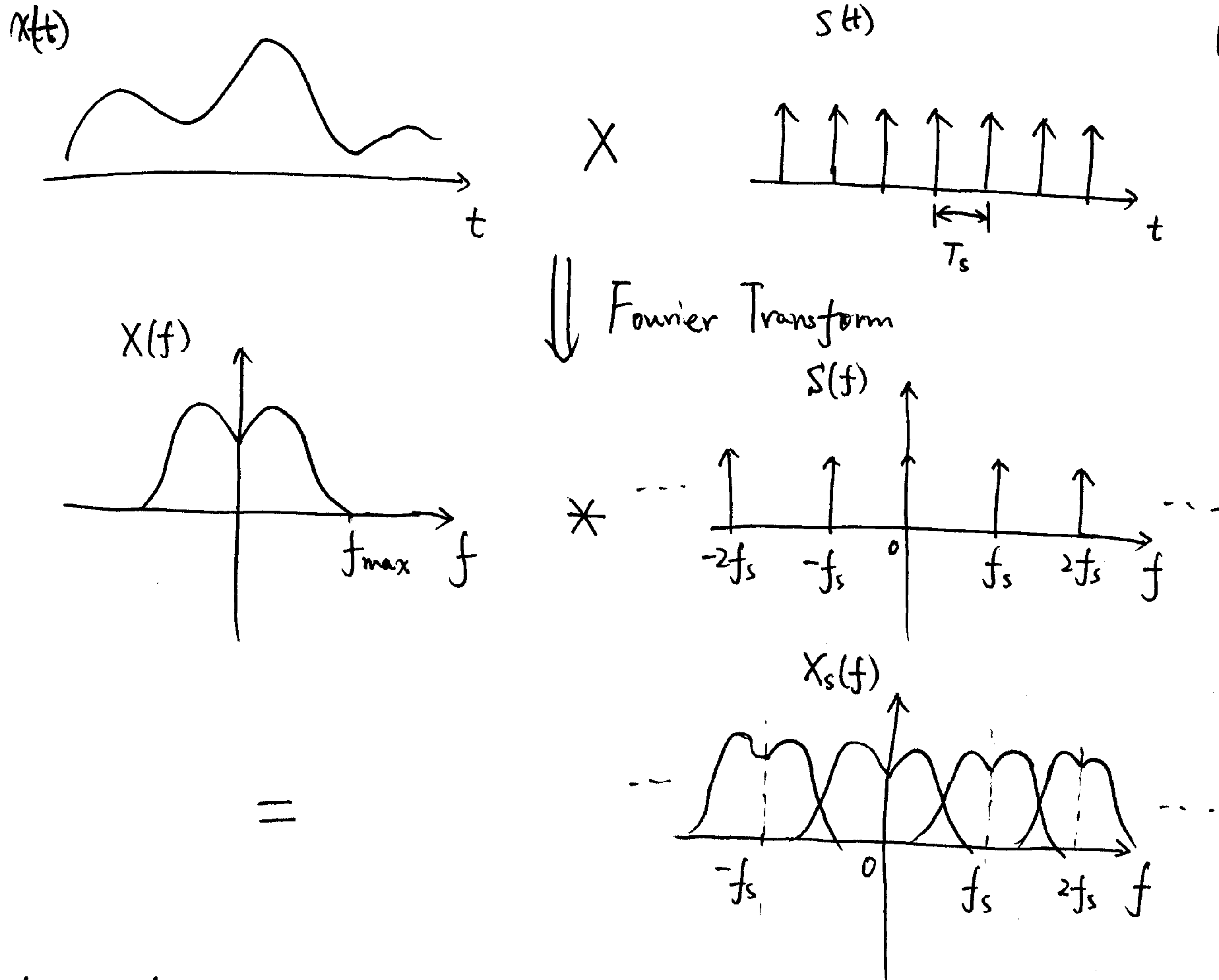
Difference between $x_s(t)$ and $x[n]$:

1) $x_s(t)$ is a continuous-time signal while $x[n]$ is discrete-time signal.

2) $x_s(t)$ takes infinite values while $x[n] = x(nT)$ takes finite values.

3) $x[n]$ is the area under $x_s(nT)$, i.e. integration.

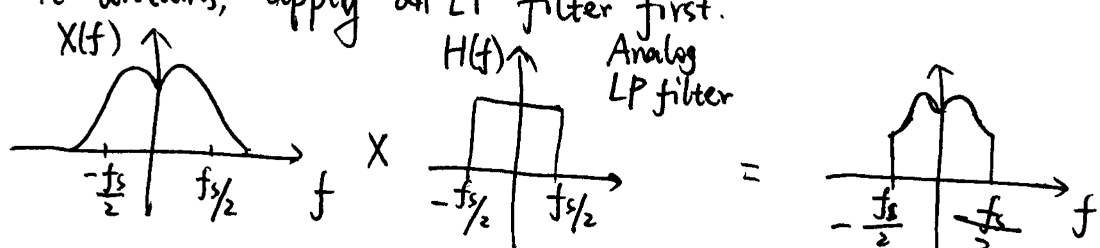
How we can analyze the frequency content of $x_s(t)$ and compare it with $x(t)$.




Therefore, if $x(t)$ contains frequency higher than $\frac{f_s}{2}$, i.e. $f_{max} > \frac{f_s}{2}$, then the replicas of $X(f)$ will overlap. This is called aliasing. When aliasing happens, we will not be able to recover $X(f)$ from $X_s(f)$, i.e. not able to recover $x(t)$ from $x_s(t)$.

How to prevent aliasing?

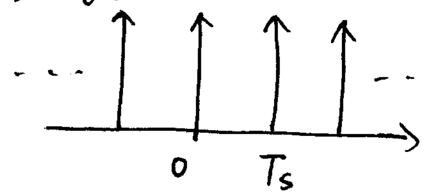
1) Make sure $x(t)$ do not contain frequencies $> \frac{f_s}{2}$, (called Nyquist freq.)
If it contains, apply an analog LP filter first.




2) Sample fast enough, i.e. Let $f_s \geq 2 f_{max}$, (called Nyquist rate)

Derivation of Fourier transform of impulse train

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$




period signal with period of $T_s$.

So it has Fourier Series expansion.

$$s(t) = \sum_{m=-\infty}^{\infty} C_m e^{j2\pi m t/T_s}$$

(a summation of periodic signals, i.e., complex sine waves)

$e^{\pm j2\pi t/T_s}$ is the fundamental sine

$e^{j2\pi t m/T_s}$ $(m = \pm 2, \pm 3, \cdots)$ are harmonic

Remember that $\int_{t_0}^{t_0+T_s} e^{j2\pi m t/T_s} \cdot e^{-j2\pi n t/T_s} \begin{cases} = 0, & \text{if } m \neq n \\ = T_s, & \text{if } m = n \end{cases}$

∴ ~~and $\int_{t_0}^{t_0+T_s} e^{j2\pi m t/T_s} = e$~~

$$\therefore\ C_m = \frac{1}{T_s}\int_{t_0}^{t_0+T_s} s(t)\, e^{-j2\pi m t/T_s}\, dt$$

$$= \frac{1}{T_s}\int_{-\frac{T_s}{2}}^{T_s/2} s(t)\, e^{-j2\pi m t/T_s}\, dt$$ (shift to another period

$$= \frac{1}{T_s}\int_{-T_s/2}^{T_s/2} \delta(t)\, e^{-j2\pi m t/T_s}\, dt$$ (only one impulse here)

$$= \frac{1}{T_s}\int_{-T_s/2}^{T_s/2} \delta(t)\, dt = \frac{1}{T_s}$$

$$\therefore\ s(t) = \sum_{n=-\infty}^{\infty} \frac{1}{T_s} e^{j2\pi m t/T_s}$$

$$\therefore\ S(f) = \int_{-\infty}^{\infty} s(t)\, e^{-j2\pi f t}\, dt = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{T_s} e^{j2\pi m t/T_s} e^{-j2\pi f t}\, dt$$

$$= \frac{1}{T_s}\sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-j2\pi(f - f_s)t}\, dt = \frac{1}{T_s}\sum_{m=-\infty}^{\infty} \delta(f - m f_s)$$