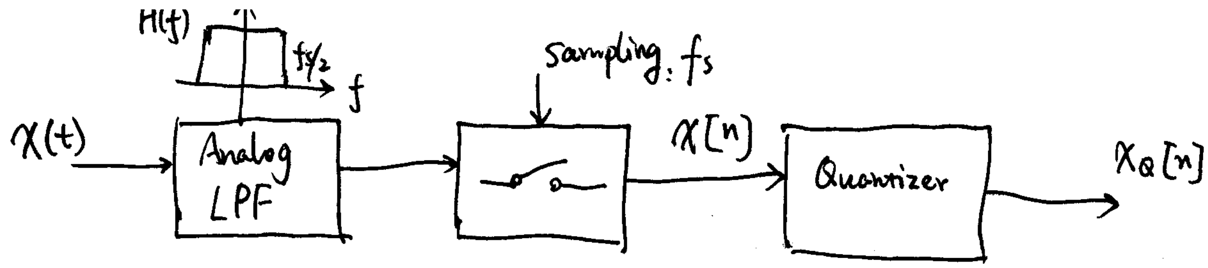
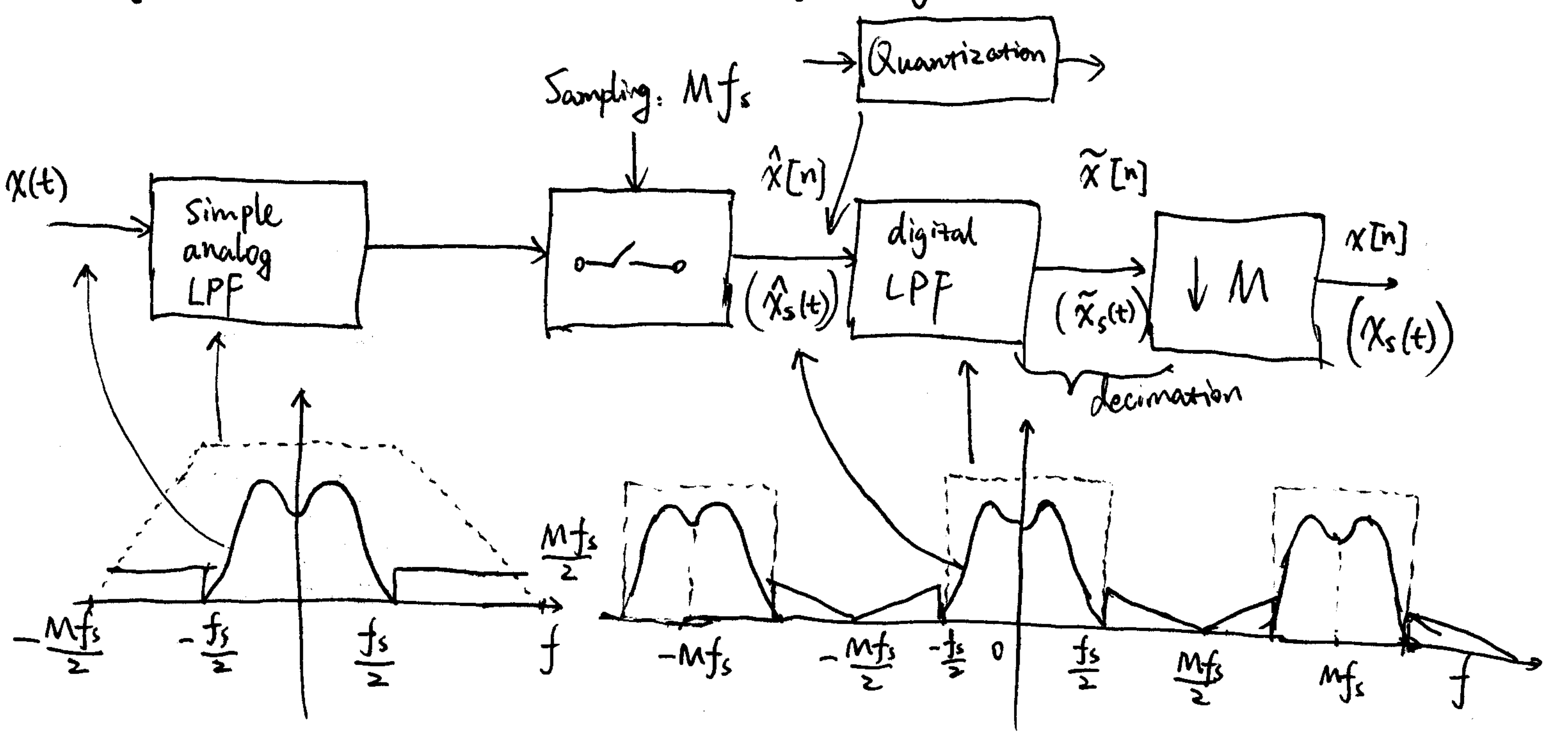


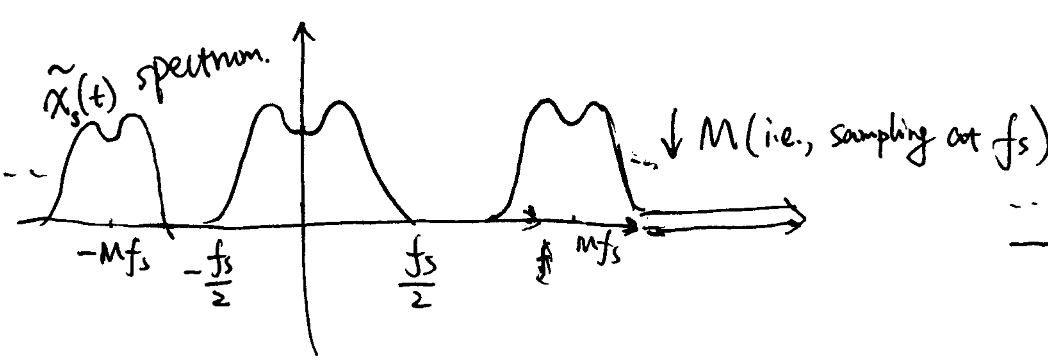
Problems: 1) Analog LPF with sharp cut-off frequency is difficult to implement. Big and expensive!

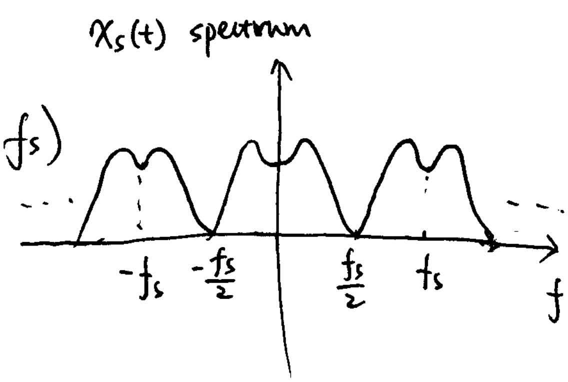
2) Multi-bi Quantizer is difficult to build.

Solution: Oversampling!

Suppose we want to retain the frequency components between 0 and $\frac{f_s}{2}$ Hz of the input $x(t)$, we can do the following:







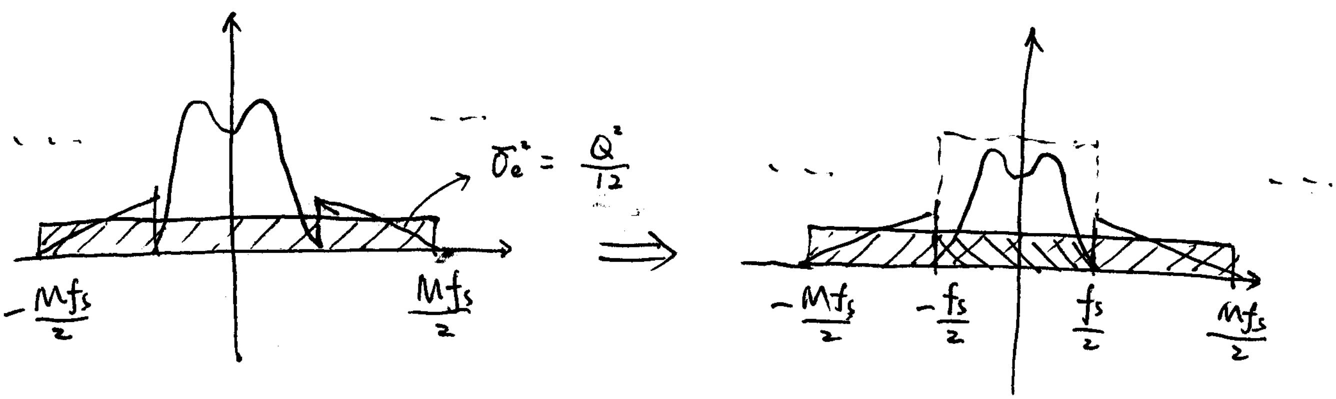
Digital LPF is much easier to implement!

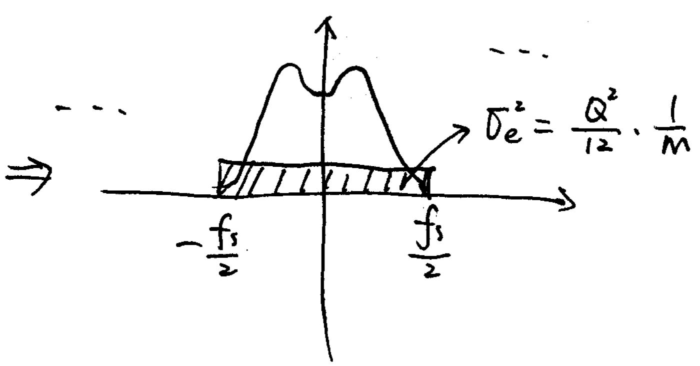
Oversampling increases quantization SNR

1) Assuming quantization noise has uniform distribution PDF, we have seen its power is $\frac{Q^2}{12}$.

2) Assuming PSD of quantization error is flat, i.e., uniformly distributed over entire frequency range, then power density is $\frac{Q^2}{12} \cdot \frac{1}{Mf_s}$

3) After digital LPF with cut-off frequency of $\frac{f_s}{2}$, only part of the quantization noise is remained, while the entire signal energe is remaine





$\therefore$ SNR increase $= 10\log_{10} M$

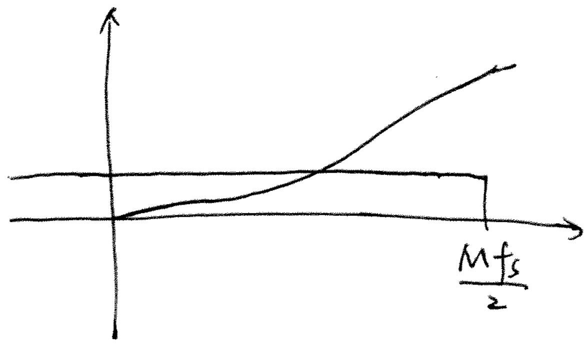
Oversampling factor M	SNR increase
2	3dB
$2^2 = 4$	6dB ← Quantization level increase 1 bit
2^3	9dB
$2^4 = 4^2$	12dB ← 2 bits

$\therefore$ Oversampling for 4^N times is equivalent to increasing quantization bit depth for N bits.

Not very efficient. Can we do better?

Idea: Shape the quantization noise spectral shape!

If we can emphasize high-frequency part of noise, ~~it'd b~~ and not change the signal.



Quantization noise: $X_Q[n] = x[n] + e[n]$

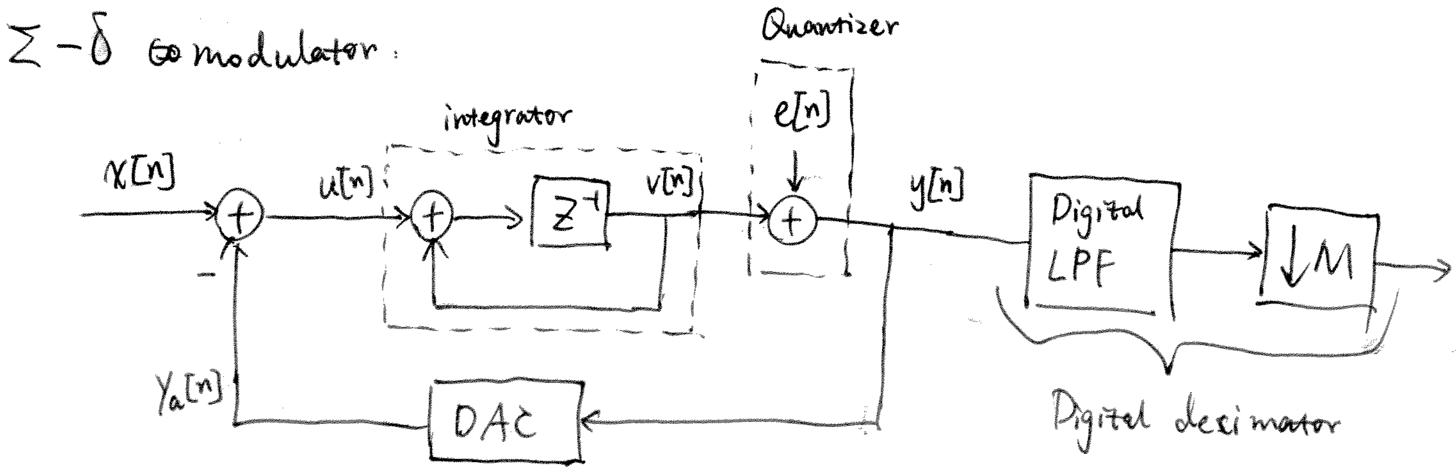
Z - transform: $Y(z) = X(z) + E(z)$

In general: $Y(z) = H_x(z) \cdot X(z) + H_E(z) E(z)$

transfer function for signal. transfer function for noise

If we can design $H_x(z)$ as all-pass filter and $H_E(z)$ as high-pass filter, it'd be great!

$\Sigma - \delta$ modulator:



Integrator: $\left(U(z) + V(z)\right) z^{-1} = V(z)$, $\therefore \frac{V(z)}{U(z)} = \frac{z^{-1}}{1-z^{-1}}$

Ideal DAC: $y_a[n] = y[n]$

Then: $\left(X(z) - Y(z)\right) \frac{z^{-1}}{1-z^{-1}} + E(z) = Y(z)$

$$\therefore X(z)z^{-1} + Y(z)z^{-1} + (1-z^{-1})E(z) = (1-z^{-1})Y(z)$$

$$\therefore Y(z) = z^{-1}X(z) + (1-z^{-1})E(z)$$

$\therefore$ $H_X(z) = z^{-1}$ Delay one sample. (All-pass).

$H_E(z) = 1-z^{-1}$ First-order differentiator (High-pass).

Time domain $y[n] = x[n-1] + e[n] - e[n-1]$

Power Density Spectrum of ~~y[n] is~~ shaped quantization noise $e_1[n] = e[n] - e[n-$

is $P_{E_1}(f) = P_E(f) \cdot |H_E(f)|^2$

or $P_{E_1}(e^{j\Omega}) = P_E(e^{j\Omega}) \cdot |H_E(e^{j\Omega})|^2 = P_E(e^{j\Omega})|1-e^{j\Omega}|^2$

$$= P_E(e^{j\Omega})\left|e^{j\frac{\Omega}{2}}\left(e^{-j\frac{\Omega}{2}} - e^{j\frac{\Omega}{2}}\right)\right|^2$$

$$= P_E(e^{j\Omega}) \cdot 4\sin^2\frac{\Omega}{2}$$

After Digital LPF with cutoff frequency of $\frac{f_s}{2}$, the rest noise power is

$$\delta_{\hat{e}_1}^2 = \int_{-\frac{f_s}{2}}^{\frac{f_s}{2}} P_{E_1}(f)\,df = P_E(f)\int_{-\frac{f_s}{2}}^{f_s/2} 4\sin^2\left(\frac{2\pi f/f_s \cdot M}{2}\right)df$$

$$= P_E(f) \cdot 2\int_0^{f_s/2} \cdot 4\sin^2\left(\pi\frac{f}{Mf_s}\right)df$$

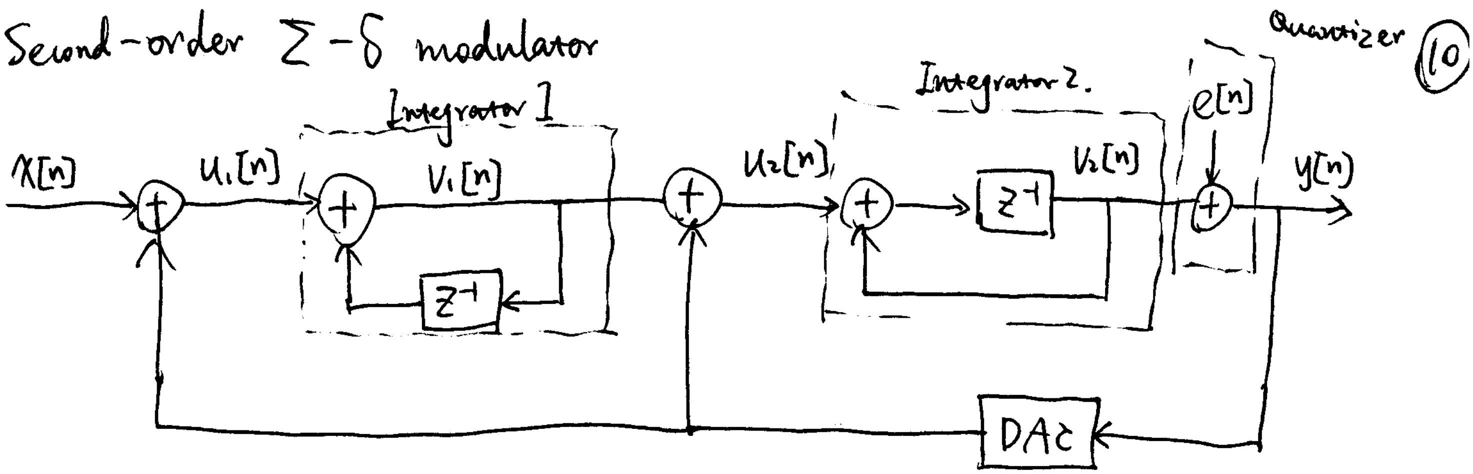
$$= \frac{Q^2}{12} \cdot \frac{\pi^2}{3} \cdot \frac{1}{M^3}$$

$\therefore$ SNR increase : $10\log_{10}(M^3) - 10\log_{10}\left(\frac{\pi^2}{3}\right)$

$$= 30\log_{10}M - 10\log_{10}\left(\frac{\pi^2}{3}\right)$$

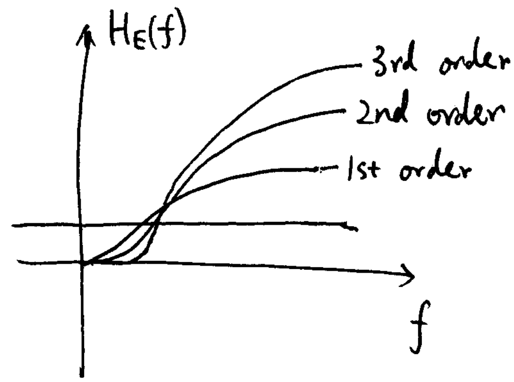
M	SNR increase	equivalent bit depth increase
2	9dB	1.5 bits
4	18dB	3 bits
⋮	⋮	⋮

Second-order Σ-δ modulator



Integrator 1: $\dfrac{V_1(z)}{U_1(z)} = \dfrac{1}{1-z^{-1}}$,

Integrator 2: $\dfrac{V_2(z)}{U_2(z)} = \dfrac{z^{-1}}{1-z^{-1}}$.



$$Y(z) = X(z)\,z^{-1} + E(z)\,(1-z^{-1})^2$$

SNR inrease: $50\log_{10} M - 10\log\left(\frac{\pi^4}{5}\right)$

∴ Doubling oversampling factor gives ~ 15dB SNR increase, i.e., about 2.5 bits quantization bit-depth increase.

L-order Σ-δ modulator: Have L-1 Integrators of $\frac{1}{1-z^{-1}}$, then 1 integrator of $\frac{z^{-1}}{1-z^{-1}}$.

Noise transfer function $H_E(z) = (1-z^{-1})^L$

signal - - - - - - $H_X(z) = z^{-1}$

SNR increase ~~20L+~~ ~~20~~ $\underline{\underline{(2L+1)\,10\log_{10} M}} - 10\log_{10}\left(\frac{\pi^{2L}}{2L+1}\right)$

Doubling oversample factor ~ SNR increase $(6L+3)$ dB ~ bit depth increase $(L+0.5)$ bits