ECE 272/472 Lecture 3: Fourier Transforms

(Continuous) Fourier Transform (FT)

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \qquad X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt$$

continuous $\longrightarrow \omega$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega \qquad x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi f t}\, df$$

continuous $\longrightarrow t$

Discrete Time Fourier Transform (DTFT)

periodic with period 2π $\longrightarrow$

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n}$$

continuous $\longrightarrow \Omega$

$$x[n] = \frac{1}{2\pi}\int_{0}^{2\pi} X(\Omega)\, e^{j\Omega n}\, d\Omega$$

discrete $\longrightarrow n$

Note: Ω is different from ω.

Question: 1) if we view $x[n]$ as $x(n$, why not taking continuous Fourier transform?

2) why ~~take only~~ integrate over only one period in inverse transform

Fourier Series

$$X[k] = \frac{1}{T}\int_{0}^{T} x(t)\, e^{-j\omega_k t}\, dt, \qquad \omega_k = \frac{2\pi k}{T}$$

discrete $\longrightarrow k$

periodic with period T. $\longrightarrow$

$$x(t) = \sum_{k=-\infty}^{\infty} X[k]\, e^{j\omega_k t}$$

continuous $\longrightarrow t$

Question:

1) why integrate over only one period in forward transfo

Discrete Fourier Transform (DFT)

periodic $\longrightarrow$

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\omega_k n}, \qquad \omega_k = \frac{2\pi k}{N}$$

discrete $\longrightarrow k$

periodic $\longrightarrow$

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, e^{j\omega_k n}$$

discrete $\longrightarrow n$

1) why sum over only one period in both forward and inverse transforms?

freq. \ time	Continuous	Discrete	
Continuous	FT	DTFT	aperiodic
Discrete	Fourier Series	DFT	periodic
	aperiodic	periodic	time \ freq.

Relation between FT and DTFT (Sampling Theorem Revisit)

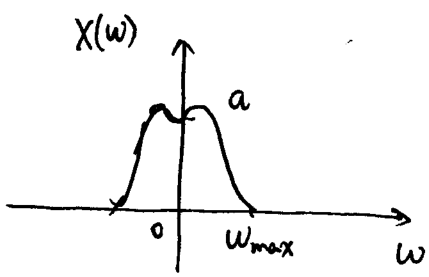
Let $x[n] = x(nT_s)$, $f_s = \frac{1}{T_s}$ Sampling rate.

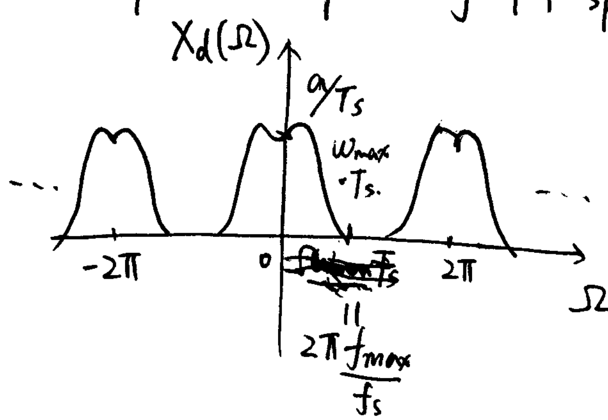
then $$X_d(\Omega) = \frac{1}{T_s} \sum_{m=-\infty}^{\infty} X\left(\frac{\Omega}{T_s} + m\frac{2\pi}{T_s}\right)$$

DTFT spectrum ($X_d(\Omega)$) FT spectrum (X)

i.e. DTFT spectrum of $x[n]$ is periodic $\underbrace{\text{replicas}}_{}$ of FT spectrum of $x(t)$.

x-axis scaled, y-axis scaled





∴ f_{max} should be less than $f_s/2$ to prevent aliasing.

Proof: Consider inverse transforms.

inverse FT: $$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(w)\, e^{jwt}\, dw$$

$$= \frac{1}{2\pi} \cancel{\int_0} \sum_{m=-\infty}^{\infty} \int_{m\cdot\frac{2\pi}{T_s}}^{(m+1)\frac{2\pi}{T_s}} X(w)\, e^{jwt}\, dw$$

$$= \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \int_{0}^{\frac{2\pi}{T_s}} X\left(w + m\frac{2\pi}{T_s}\right) e^{j\left(w + m\frac{2\pi}{T_s}\right)t}\, dw$$

$$= \frac{1}{2\pi} \cancel{\sum_{m=-\infty}^{\infty}} \int_{0}^{\frac{2\pi}{T_s}} e^{jwt} \sum_{m=-\infty}^{\infty} X\left(w + m\frac{2\pi}{T_s}\right) e^{jm\frac{2\pi}{T_s}t}\, dw$$

$$\therefore\ x[n] = x(nT_s) = \frac{1}{2\pi}\int_{0}^{\frac{2\pi}{T_s}} e^{jwnT_s} \sum_{m=-\infty}^{\infty} X\left(w + m\frac{2\pi}{T_s}\right) \underbrace{e^{jm\frac{2\pi}{T_s}\cdot nT_s}}_{=1}\, dw$$

Let $\Omega = w\cdot T_s$ $$= \frac{1}{2\pi}\int_{0}^{2\pi} e^{j\Omega n} \cdot \cancel{\frac{1}{T_s}} \sum_{m=-\infty}^{\infty} X\left(\frac{\Omega}{T_s} + m\frac{2\pi}{T_s}\right)\cdot \frac{1}{T_s}\, d\Omega$$

By definition, $$x[n] = \frac{1}{2\pi}\int_{0}^{2\pi} X_d(\Omega)\, e^{j\Omega n}\, d\Omega$$

Comparing the above two equations, we have

$$X_d(\Omega) = \frac{1}{T_s}\sum_{m=-\infty}^{\infty} X\left(\frac{\Omega}{T_s} + m\frac{2\pi}{T_s}\right) \qquad \square$$

Therefore, for simplicity, we only specify one period of $X_d(\Omega)$, i.e. $-\pi \le \Omega < \pi$, or $0 \le \Omega < 2\pi$.

In computers, everything is discrete, including Fourier spectra.

How to discretize DTFT spectrum $X_d(\Omega)$?

Suppose we have $x[n]$ that only has non-zero values ~~between~~ for $0 \le n < N$.

Taking DTFT, we have

$$X_d(\Omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n} = \sum_{n=0}^{N-1} x[n]\, e^{-j\Omega n}, \quad \text{periodic } 2\pi$$

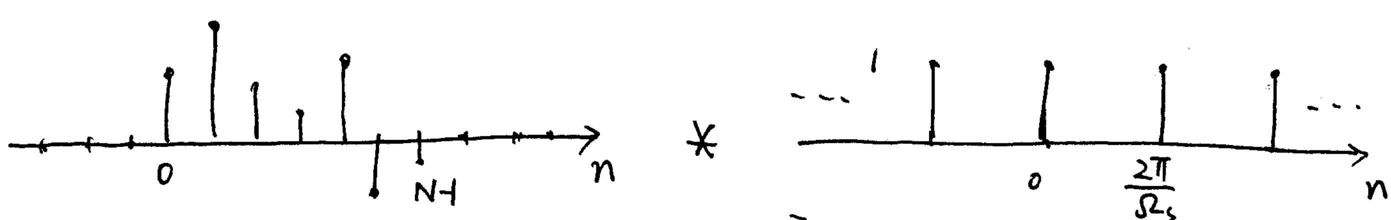
How should we sample $X_d(\Omega)$ so that $x[n]$ can be recovered from the samples? Another application of Sampling Theorem!

Sampling $X_d(\Omega)$ with period Ω_s $\Longleftrightarrow$ $X_d(\Omega) \cdot \delta(\Omega - k\Omega_s)$

$$\Big\Downarrow \text{ inverse DTFT}$$

$$\cancel{x[n] * \delta[} \quad x[n] * \delta\left[n - m\frac{2\pi}{\Omega_s}\right]$$

$$\cancel{x(nT) * \delta(}$$



To prevent aliasing, we need $\frac{2\pi}{\Omega_s} \geq N$, i.e., $\Omega_s \leq \frac{2\pi}{N}$, i.e., we should use at least N samples to represent $X_d(\Omega)$ for $0 \le \Omega < 2\pi$

If we use N samples, then the frequencies of these samples are $\frac{2\pi}{N} \cdot k = k\Omega_s$ for $0 \le k \le N-1$. We can define discrete spectrum

DFT: $$X[k] = X_d(k\Omega_s) = \sum_{n=0}^{N-1} x[n]\, e^{-jk\Omega_s n} = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi kn}{N}} \quad \text{(periodic, } N\text{)}.$$

IDFT: $$\hat{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi kn}{N}} \quad \text{(periodic, } N\text{)}$$

when $0 \le n \le N-1$, $\hat{x}[n] = x[n]$. And we do not use $\hat{x}[n]$ for $n \notin [0, N-1]$ anyway, so DFT pair is defined for $0 \le n, k \le N-1$.

We have discussed the relation between FT, DTFT, DFT, and Fourier Series

FT

0	$f_s/2$	f_s	f	: frequency
0	πf_s	$2\pi f_s$	ω	: angular freq.

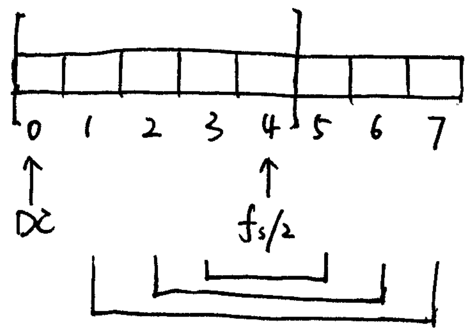
DTFT

0	$1/2$	1	f'	: normalized frequency
0	π	2π	Ω	------ angular freq.

DFT

0	$\frac{N}{2}+1$	N	n	when N is even

e.g., $N=8$



conjugate symmetric since time-domain signal is real.

We only need half-spectrum, i.e., $0 \le n \le \frac{N}{2}+1$, for audio signals.

Fast Fourier Transform (FFT) is an efficient algorithm to calculate DFT

Complexity: DFT: $O(N^2)$ N^2 complex multiplications

FFT: $O(N\log N)$, "divide-and-conquer" Advanced Algorithms

Applications. convolution: $x[n] * y[n]$ naive calculation: $O(N^2)$

first discovered by (Gauss, 1805)

re-discovered by (Cooley, Tukey 1965.)

using FFT: $\underset{N\log N}{\mathrm{IFFT}}\left(\underset{N\log N}{\mathrm{FFT}(x)} \underset{N}{\times} \underset{N\log N}{\mathrm{FFT}(y)}\right)$

$\therefore$ $3N\log N + N$ complex multiplications. $O(N\log N)$.

correlation, Auto-correlation, etc.

~~No. 1~~ One of Top 10 Algorithms in 20th century!