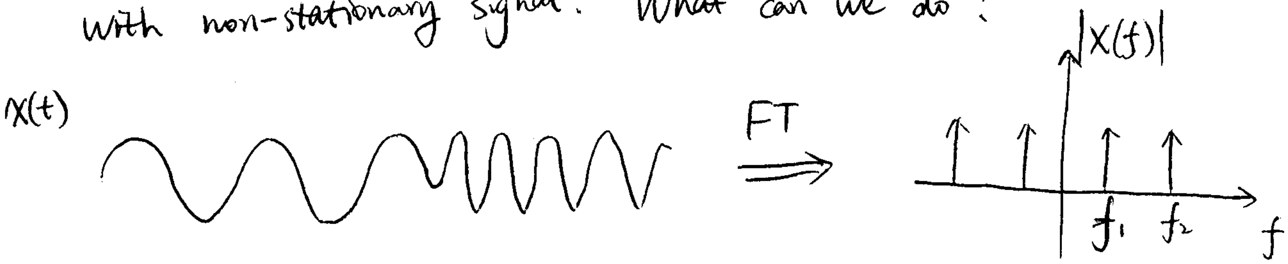
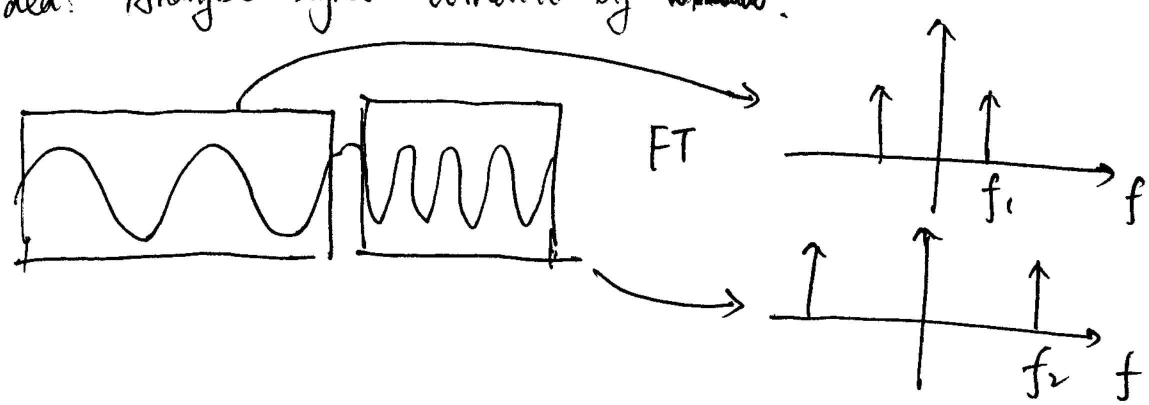
Frequency spectrum shows the <u>aggregated</u> frequency components of the entire signal of analysis. We don't when does each freq. component happen in the signal. But this information is important if we are dealing with non-stationary signal. What can we do?



Idea: Analyze signal window by ~~window~~.



Time-frequency resolution tradeoff.

Δt: time resolution: Length of window, because freq. components of signal within the window cannot be resolved.

Δf: freq. resolution: (without zero-padding) freq. difference between adjacent bins.

Suppose $x[n]$ was sampled from $x(t)$ with f_s.

Take a window of M samples, then $\Delta t = \frac{M}{f_s}$ (second)

After DFT, we have M samples to represent the spectrum from 0 to f_s Hz

$$\therefore \Delta f = \frac{f_s}{M} \text{ (Hz)}$$

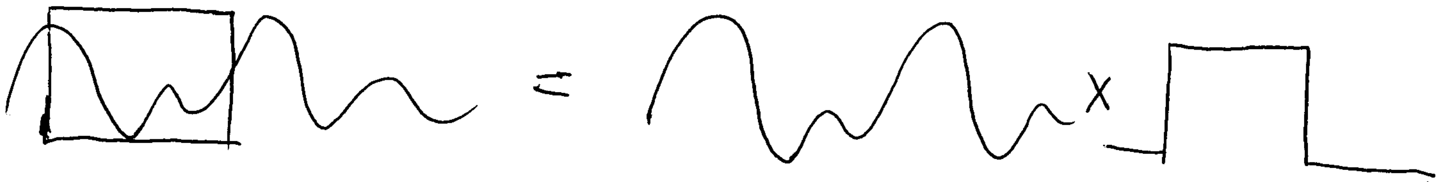
$$\therefore \Delta t \cdot \Delta f = 1$$

Therefore, to estimate Δf, the window length in second Δt is the only critical ~~or~~ variable. f_s and M don't matter!

e.g. if $\Delta t = 100$ ms, $\Delta f = 10$ Hz.

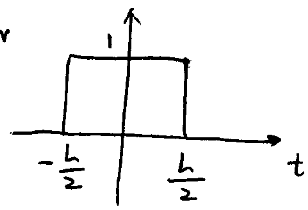
Effect of Windowing

Taking a chunk of $x(t)$ for analysis is equivalent to multiplying $x(t)$ with a rectangular window function.



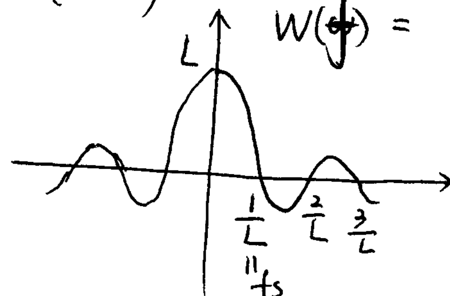
$$FT(x(t) \times w(t)) = FT(x(t)) * FT(w(t))$$

rectangular window of Length L second



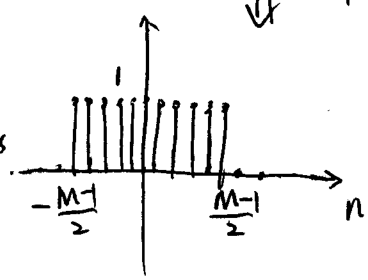
FT $\Longrightarrow$

$$W(f) = \cancel{\frac{\sin(\pi \omega L)}{\pi \omega}} \quad \frac{\sin(\pi f L)}{\pi f}$$



$\Downarrow$ sample f_s

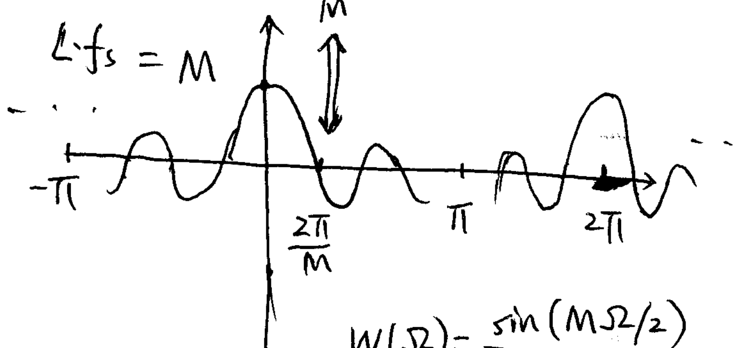
Length M samples



$M = L \cdot f_s$

DTFT $\Longrightarrow$

$L \cdot f_s = M$



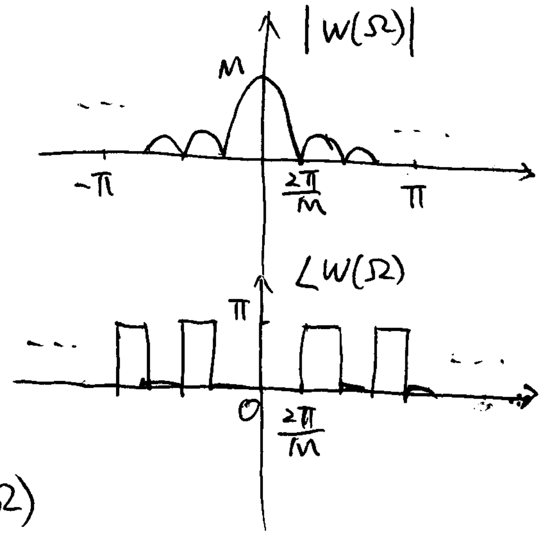
$$W(\Omega) = \frac{\sin(M\Omega/2)}{\sin(\Omega/2)}$$

(aliased sinc function, asinc function)

symmetric windows (zero-phase windows)

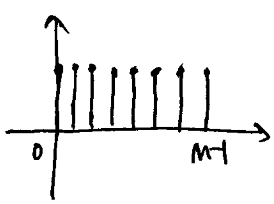
$\Longleftrightarrow$ spectrum is real

$\Longleftrightarrow$ phase spectrum only take 0 or π values



Shift window in time

$\Longleftrightarrow$ multiply a phase linear in spectrum



DTFT $\Longrightarrow$ $e^{-j\Omega(M-1)/2} \cdot W(\Omega)$

zero-phase spectrum

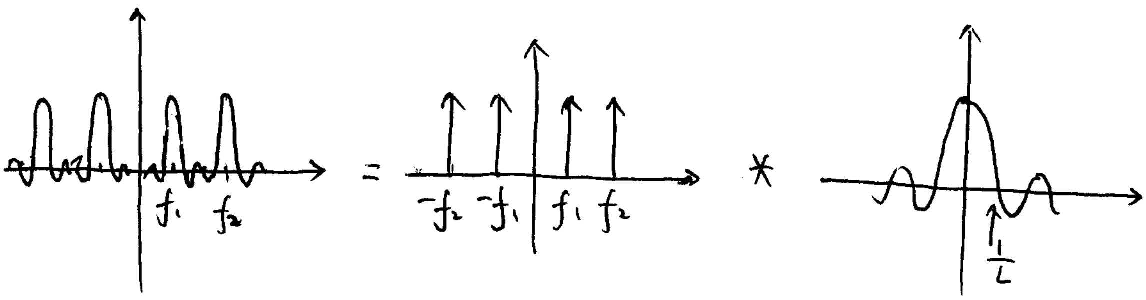
Causal window

Windowing smears (smoothes) spectrum, i.e., reduces frequency resolution.

Ideal frequency resolution $\Delta f = \frac{1}{\Delta t}$

Considering window effect: Let $x(t) = \sin(2\pi f_1 t) + \sin(2\pi f_2 t)$ (in phase sinusoids).

$$FT(x(t) \cdot w(t)) = FT(x(t)) * FT(w(t))$$

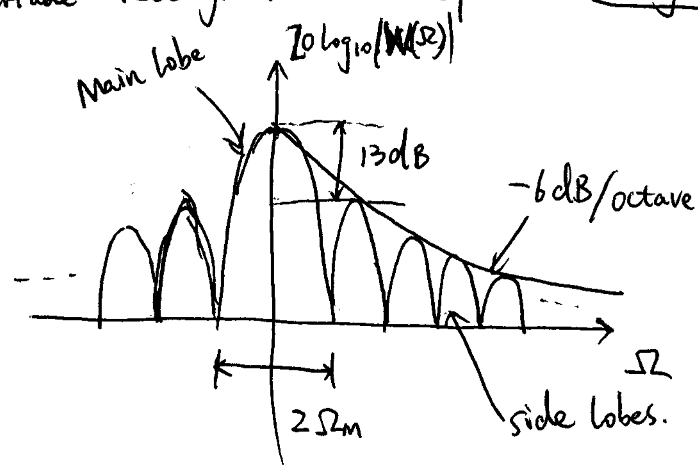


To make sure the two sinusoids are resolved, we need $|f_1 - f_2| > \frac{2}{L}$, i.e.

$$\Delta f > \frac{2}{\Delta t}$$

Clearly, the real frequency resolution is affected by the shape of window spectrum.

A log-amplitude rectangular window spectrum magnitude



$\Omega_M = \frac{2\pi}{M}$

translating to frequency in Hz

$f = \frac{f_s}{M} = \frac{1}{L}$ ← length in second.

We want 1) narrow main lobe
2) low side lobes.

Rect window is good at ①, but not good at ②.

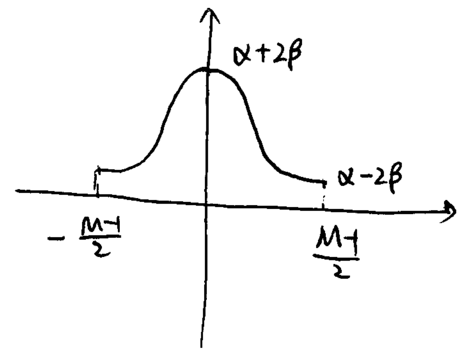
Can we design other windows to achieve these?

Reason why Rect window has high side-lobe is because of the abrupt change at the window boundary.

Idea: Smooth out the boundary.

Generalized Hamming Window Family.

$$W_H[n] = W_R[n]\left[\alpha + 2\beta\cos\left(\frac{2\pi n}{M}\right)\right]$$



- Hann window : $\alpha = 1/2$, $\beta = 1/4$.

$$W_H[n] = W_R[n]\left[\frac{1}{2} + \frac{1}{2}\cos\left(\frac{2\pi n}{M}\right)\right] = W_R[n]\cdot\cos^2\left(\frac{\pi}{M}n\right)$$

Main lobe width : $4\Omega_M$

Remember $\Omega_M = \frac{2\pi}{M}$

First side lobe height : -31dB

Side lobe roll off : -18dB/octave

- Hamming window : $\alpha = 0.54$. $\beta = 0.23$

Main lobe width : $4\Omega_M$

First side lobe height : -41dB

Sidelobe roll off : -6dB/octave.

$\Delta f > \frac{4}{\Delta t}$ for Hann and Hamming windows, which is worse than rectangular window. But because sidelobes are significantly reduced, they are often used in practice.

Spectral Interpolation, Zero padding.

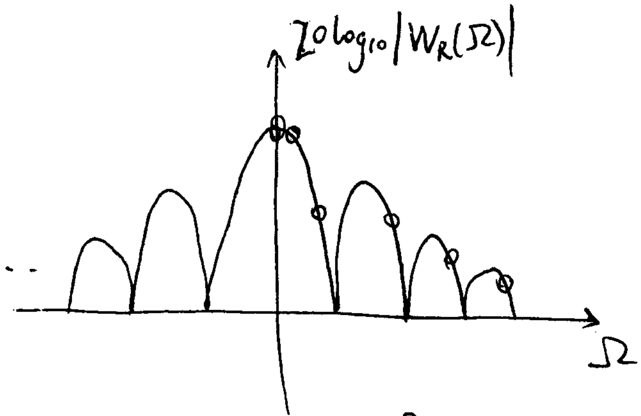
~~In DFT~~, suppose we take an M-point windowed signal and perform DFT to get its spectrum, its spectrum is represented by M points as well. This is often not enough to see the detailed content.

Analysis: M points to represent 2π. $\Leftrightarrow$ each point for $2\pi/M = \Omega_M$

$\Rightarrow$ we only use 2 points to represent the main lobe of a Rect window.

- - - - - - 4 points - - - - - - - - - - Hamming/Hann - -

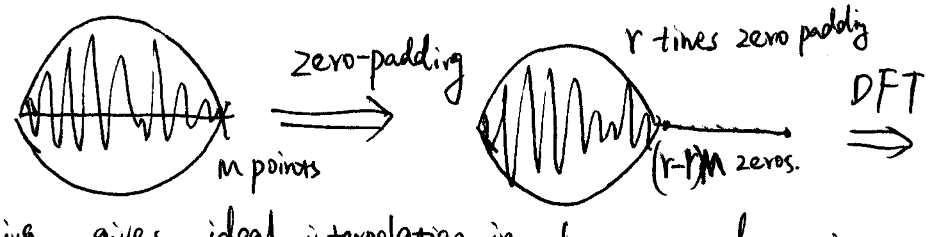
- - - - - - 1 point - - - - - - - - side lobe . . these windows.



This is clearly not enough to show the shape of the spectrum.

Can we use more points ?

Zero padding: append zeros to windowed signal before perform DFT.



Zero padding gives ideal interpolation in frequency domain

Proof: consider ~~M-point~~ N-point $x[n]$ with DFT spectrum $X[k]$.

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi kn}{N}} \qquad \text{(IDFT)}$$

Let's zero pad $x[n]$ with infinitely many zeros on both sides, then take DTFT to look at its spectrum.

$$X(\Omega) = \sum_{n=-N/2}^{N/2} x[n]\, e^{-j\Omega n} = \sum_{n=-\frac{N}{2}}^{N/2}\left[\frac{1}{N}\sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi kn}{N}}\right] e^{-j\Omega n}$$

$$= \sum_{k=0}^{N-1} X[k]\left[\frac{1}{N}\sum_{n=-\frac{N}{2}}^{N/2} e^{j\left(\frac{2\pi k}{N}-\Omega\right)n}\right]$$

$$= \sum_{k=0}^{N-1} X[k]\, \mathrm{asinc}_N\!\left(\Omega - \frac{2\pi k}{N}\right) \qquad \mathrm{asinc}_N(\Omega) = \frac{\sin(N\Omega/2)}{N\sin(\Omega/2)}$$

This means $X(\Omega)$ can be interpolated from $X[k]$ using infinite zero padding, and the interpolation is ideal (as it uses asinc function!)

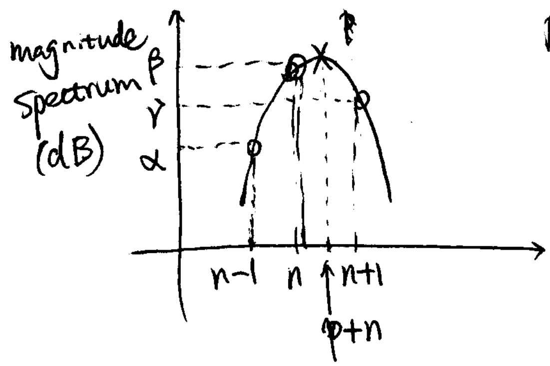
Note: After zero-padding, we are using more points to represent the spectrum. i.e., Δf between adjacent bins is smaller.

But, frequency resolution is Not Changed!

Sinusoids that ~~are~~ were not resolved are still not resolved.

△ Zero padding does not increase frequency resolution!

Quadradic Peak Interpolation



Real Peak may lie in between two frequency bins.

We could solve this by large number of zero padding, but it's computationally expensive.

Another sub-optimal solution: fit neighboring bins with a parabola.

$$y(x) = \cancel{a(x-p)^2+b} \quad a\left[x-(p+n)\right]^2 + b$$

$$y(n-1) = a(p+1)^2 + b = \alpha$$
$$y(n) = ap^2 + b = \beta$$
$$y(n+1) = a(p-1)^2 + b = \gamma$$

$$\Rightarrow \quad p = \frac{1}{2}\,\frac{\alpha-\gamma}{\alpha-2\beta+\gamma} \quad \in \left[-\tfrac{1}{2}, \tfrac{1}{2}\right].$$

$$y(p+n) = \beta - \frac{1}{4}(\alpha-\gamma)p$$

Quadratically Interpolated FFT procedure. for spectral peak detection.

1. Make sure window is long enough to resolve all sinusoids.
2. Use enough zero padding (e.g., 4 times) to oversample spectrum.
3. Use quadratic interpolation of three samples around a peak.
4. solve parabola equations to get intepolated amplitude and freq.

(optional) 5. Use quadratic or linear interpolation on the unwrapped phase of the peak