ECE 272/472: Lecture 7: Digital Filters

System: (deterministic) $\xrightarrow{x[n]} \boxed{T} \longrightarrow y[n]$ Denote $y[n] = T\{x[n]\}$

Linear Systems: iff $T\{x_1[n] + x_2[n]\} = T\{x_1[n]\} + T\{x_2[n]\}$ additivity

and $T\{a\,x[n]\} = a\,T\{x[n]\}, \ \forall a$ scaling

$$\Leftrightarrow \quad T\{a\,x_1[n] + b\,x_2[n]\} = a\,T\{x_1[n]\} + b\,T\{x_2[n]\}, \ \forall a, b.$$

Time-Invariant Systems: (or shift-invariant systems)

For any input sequence $x[n]$ and its time-shifted version $x_1[n] = x[n-n_0], \forall n$
if $T\{x_1[n]\} = y[n-n_0]$, where $T\{x[n]\} = y[n]$, then the system is time-invariant.

In other words, if the input is delayed, then the output is delayed for the same amount, but the shape remains.

(Note: Homework grading system is not time-invariant!)

Linear Time-Invariant (LTI) Systems:

Both linear and time-invariant.

Write $x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$

(linearity)

$$\therefore \ y[n] = T\{x[n]\} = T\Big\{\sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]\Big\} = \sum_{k=-\infty}^{\infty} x[k]\,T\{\delta[n-k]\}$$

(time-invariant) $$= \sum_{k=-\infty}^{\infty} x[k]\,h[n-k], \quad \text{where } h[n] = T\{\delta[n]\}$$

$$= x[n] * h[n]$$

$\therefore$ An LTI System is <u>completely</u> characterized by its impulse response!

Causality: A system is <u>causal</u> if $\forall n_0$, the output value $y[n_0]$ only depends on current and pevious input values $x[n]$ where $n \le n_0$.

Stability: A system is <u>stable</u> if for every bounded input, the output is bounded, i.e., $|x[n]| \le B_x < \infty$ for all n . $\exists B_x, B_y$.

$$\Rightarrow |y[n]| \le B_y < \infty, \text{ for all } n.$$

For LTI Systems: It is stable iff the impulse response is absolutely summable: i.e., $B_h = \sum_{k=-\infty}^{\infty} |h[k]| < \infty$ (not hard to prove).

For LTI Systems: Causal iff impulse response $h[n] = 0$, $\forall n < 0$

Frequeny Response:
$$y[n] = x[n] * h[n]$$
$$= \sum_{k=-\infty}^{\infty} h[k] \cdot x[n-k]$$

If we Let $x[n] = e^{jwn}$, i.e, complex sinusoidal input sequence.

$$\text{then } y[n] = \sum_{k=-\infty}^{\infty} h[k]\, e^{jw(n-k)}$$
$$= \sum_{k=-\infty}^{\infty} h[k]\, e^{-jwk} \cdot e^{jwn}$$
$$= H(e^{jw}) \cdot e^{jwn} = H(e^{jw}) \cdot x[n]$$

$H(e^{jw})$ is the frequeny response of the system

$$H(e^{jw}) = |H(e^{jw})| \cdot \cancel{\measuredangle}\, e^{j\angle H(e^{jw})}$$

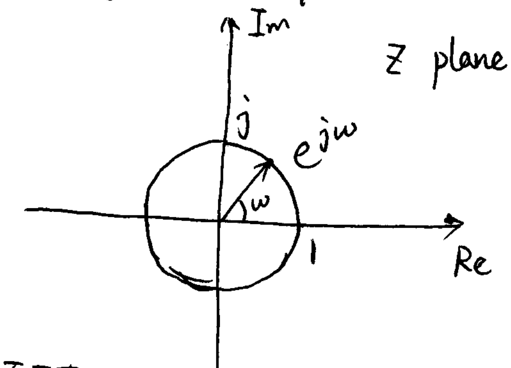
One way to get $H(e^{jw})$ for all w: try all kinds of complex sinusoidal inputs with different frequencies, and calculate the difference between output and input.

Another way: Perform DTFT on impulse response $h[n]$.

DTFT: $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$

Z transform: $X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$ (Replace $e^{j\omega}$ with z)

This gives us the entire complex plane to analyze the system instead of just the unit circle.



The Z transform equation does not always converge. Depending on $x[n]$, it has a Region of Convergence (ROC).

If ROC contains the unit circle, then DTFT converges (exists).

Properties of Z transforms (similar to those of Fourier transforms)

- Linearity: $a x_1[n] + b x_2[n] \overset{Z}{\longleftrightarrow} a X_1(z) + b X_2(z)$, ROC contains $R_{x_1} \cap R_{x_2}$
- time shifting: $x[n-n_0] \overset{Z}{\longleftrightarrow} z^{-n_0} X(z)$ ROC $= R_x$
- Convolution: $x_1[n] * x_2[n] \overset{Z}{\longleftrightarrow} X_1(z) X_2(z)$, ROC contains $R_{x_1} \cap R_{x_2}$

Z transform of LTI Systems.

$$y[n] = \cancel{x[n]}\ h[n] * x[n]$$

$$Y(z) = H(z) \cdot X(z)$$

System function

Since $h[n] \overset{Z}{\longleftrightarrow} H(z)$, ROC form a unique pair, $H(z)$ completely characterizes the LTI system.

If ROC contains the unit circle, then $H(e^{j\omega})$ exists, and it also completely characterizes the system.

A general Causal LTI filter represented by difference equation.

$$\sum_{k=0}^{N} a_k \, y[n-k] = \sum_{l=0}^{M} b_l \, x[n-l]$$

Take Z transform: $\displaystyle\sum_{k=0}^{N} a_k z^{-k} \cdot Y(z) = \sum_{l=0}^{M} b_l z^{-l} X(z)$

$$\therefore \; H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{l=0}^{M} b_l z^{-l}}{\sum_{k=0}^{N} a_k z^{-k}}$$

$\leftarrow$ polynomial of order M. (degree)

(rewritten as)

$$= \frac{b_0}{a_0} \frac{\prod_{l=1}^{M} (1 - c_l z^{-1})}{\prod_{k=1}^{N} (1 - d_k z^{-1})}$$

$\leftarrow$ M complex roots.

[Fundamental Theorem of Algebra: Any single variable polynomial of order M has exactly M complex roots, counted with multiplicity.].

Also, any non-real root must have its complex conjugate ~~ta~~ also being a root.

$\therefore$ every term $(1 - c_l z^{-1})$ in the numerator contributes ~~to~~ a zero at $z = c_l$ and a pole at $z = 0$.

every term $(1 - d_k z^{-1})$ in the denominator contributes a pole at $z = d_k$ and a zero at $z = 0$

IIR filter: $h[n]$ has infinitely many nonzeros.

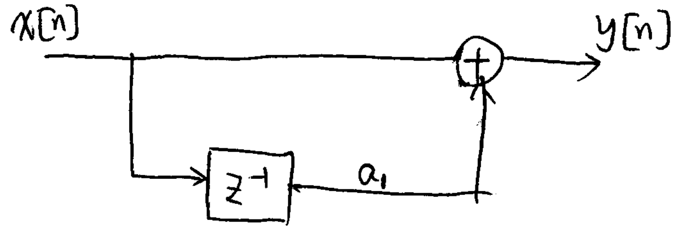
If $H(z)$ contains at least one nonzero pole, then IIR.

FIR filter: $h[n]$ has finite nonzeros.

~~If all the poles of H(z) are zeros~~

If $H(z)$ has no pole other than $z=0$, then FIR.

A simple FIR filter



n	x[n] ($\delta[n]$)	y[n]
0	1	1
1	0	a_1
2	0	0
3	0	0
⋮	⋮	⋮

$\therefore h[n] = \delta[n] + a_1 \delta[n-1]$

$$y[n] = x[n] + a_1 x[n-1]$$

$$Y(z) = X(z) + a_1 z^{-1} X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = 1 + a_1 z^{-1} = \frac{z+a_1}{z}$$

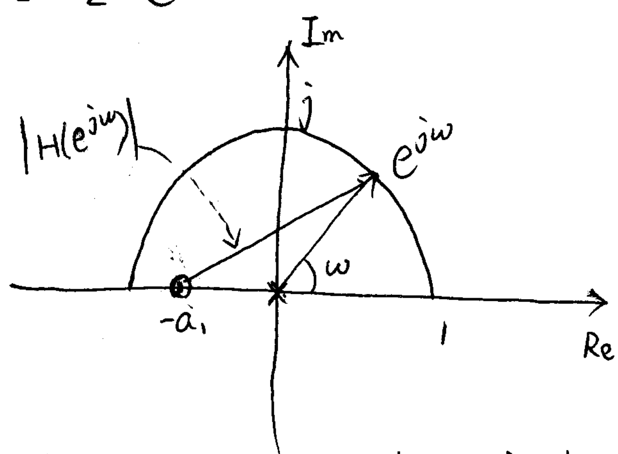
1 - zero : $z = -a_1$, 1 pole : $z = 0$. FIR filter

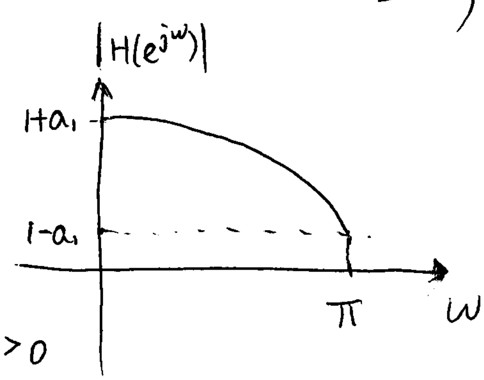
$$|H(z)| = \left|\frac{z+a_1}{z}\right|$$

Frequency response : $|H(e^{j\omega})| = \left|\frac{e^{j\omega}+a_1}{e^{j\omega}}\right| = |e^{j\omega} - (-a_1)|$

Let $z = e^{j\omega}$

(distance between $e^{j\omega}$ and zero.)



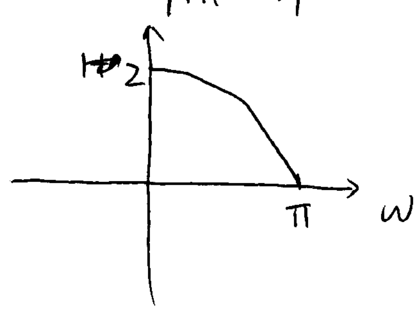


if $a_1 > 0$

$\therefore$ if $\omega = 0$, $|H(e^{j\omega})| = 1 + a_1$

$\cdots$ $\omega = \pi$, $|H(e^{j\omega})| = 1 - a_1$

Low-pass filter.

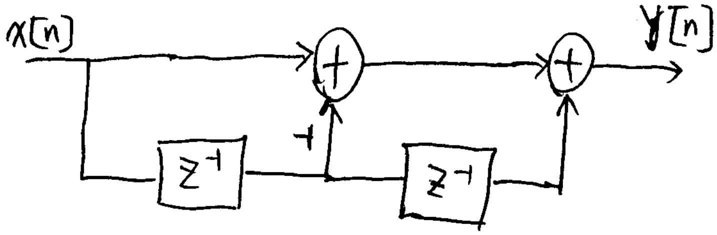


If $a_1 = 1$, then $|H(e^{j\omega})| = 0$ when $\omega = \pi$

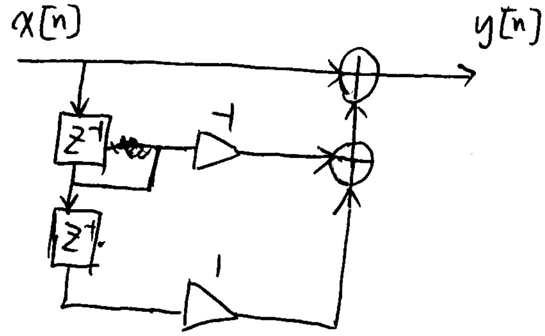
If $a_1 = 0$, then $|H(e^{j\omega})| \equiv 1$, All pass filter

If $a_1 < 0$, then high-pass filter

A second-order FIR filter



or



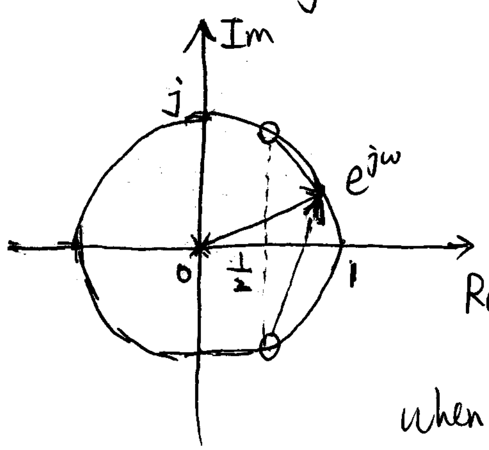
$$y[n] = x[n] - x[n-1] + x[n-2]$$

$$Y(z) = X(z) - z^{-1}X(z) + z^{-2}X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = 1 - z^{-1} + z^{-2} = \frac{z^2 - z + 1}{z^2}$$

poles: $z = 0$ twice.

zeros: roots of $z^2 - z + 1 = 0 \Rightarrow z = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1}{2} \pm j\frac{\sqrt{3}}{2} = e^{\pm j\frac{\pi}{3}}$



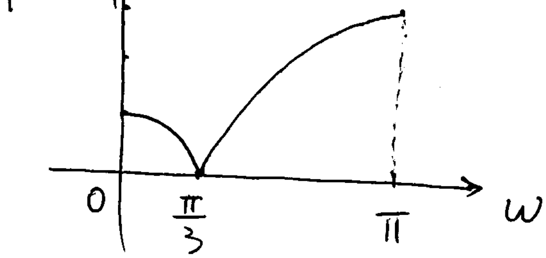
$$H(z) = \frac{z^2 - z + 1}{z^2} = \frac{(z - e^{j\frac{\pi}{3}})(z - e^{-j\frac{\pi}{3}})}{z^2}$$

$$|H(e^{j\omega})| = |e^{j\omega} - e^{j\frac{\pi}{3}}|\,|e^{j\omega} - e^{-j\frac{\pi}{3}}|$$

when $\omega = 0$, $|H(e^{j\omega})| = 1\cdot 1 = 1$

$\omega = \frac{\pi}{3}$, $\cdots = 0$

$\omega = \pi$, $|H(e^{j\omega})| = \sqrt{3}\cdot\sqrt{3} = 3$



In general, LTI causal FIR filter:

$$y[n] = \sum_{l=0}^{M} b_l \, x[n-l]$$

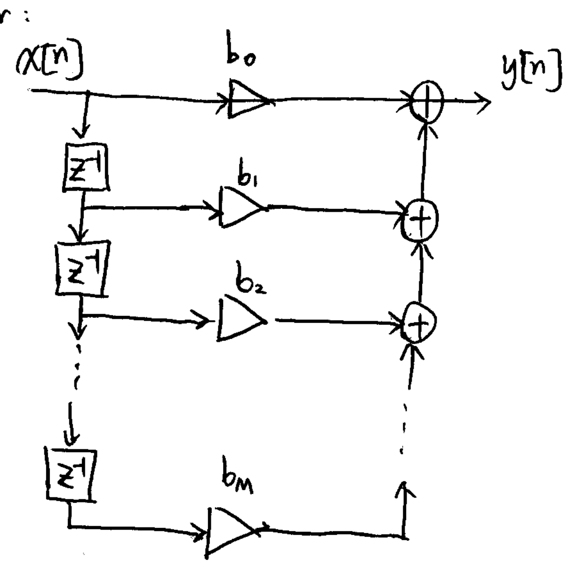
order: M (taps).

zeros: M zeros

poles: M poles all at $z=0$.

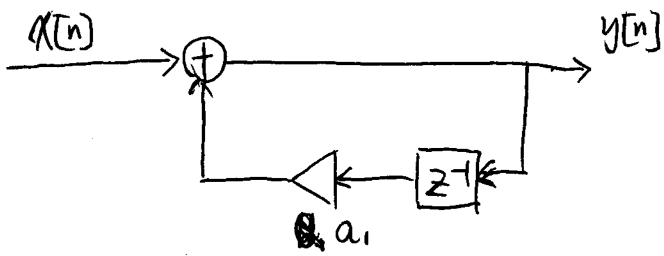
impulse response:

$$h[l] = \begin{cases} b_l, & 0 \le l \le M \\ 0, & \text{otherwise} \end{cases}$$



feed-forward coefficients.

A simple filter with a feedback loop:



$$y[n] = x[n] + a_1 y[n-1]$$

impulse response: Let $x[n] = \delta[n]$, initialize $y[-1] = 0$

n	x	y
0	1	1
1	0	a_1
2	0	a_1^2
3	0	a_1^3
⋮	⋮	⋮

∴ $y[n]$ decays exponentially, but goes on foreve

IIR filter

How long does it take for $y[n]$ to decay less than quantization noise level?

~~Supp~~ $a_1^k < 2^{1-N}/2$

$\cancel{k < (1-N)\log_{a_1} 2}$

$k \log_2 a_1 < 1-N$

$k > \dfrac{1-N}{\log_2 a_1}$

if $N = 16$ bit

$a_1 = 0.9$

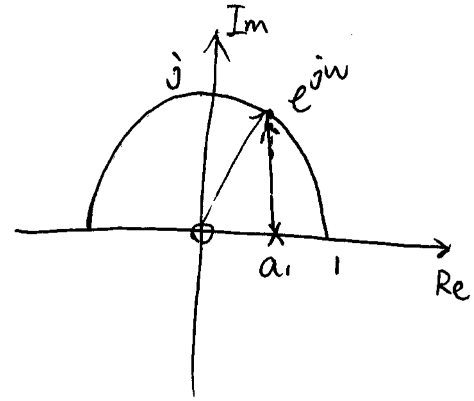
then $k >$ ~~98.7~~ 105.3

Apply z transform: $Y(z) = X(z) + a_1 z^{-1} Y(z)$

$$\therefore \quad H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - a_1 z^{-1}} = \frac{z}{z - a_1}$$

One zero at $z=0$; one pole at $z = a_1$

Frequency response $H(e^{jw}) = \dfrac{e^{jw}}{e^{jw} - a_1}$

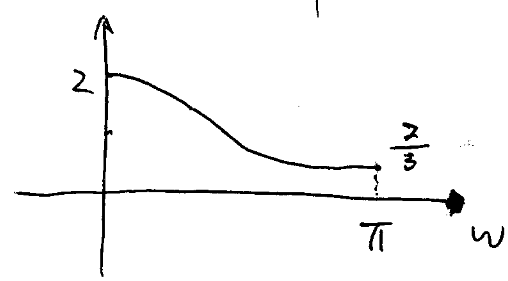


$$|H(e^{jw})| = \frac{1}{|e^{jw} - a_1|}$$

when $w=0$, $|H(e^{jw})| = \dfrac{1}{|1-a_1|}$

$w = \pi$ ---- $= \dfrac{1}{|1+a_1|}$

if $a_1 = 0.5$,



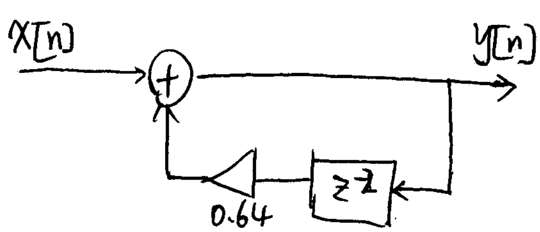
Clearly, if $0 < a_1 < 1$, LPF

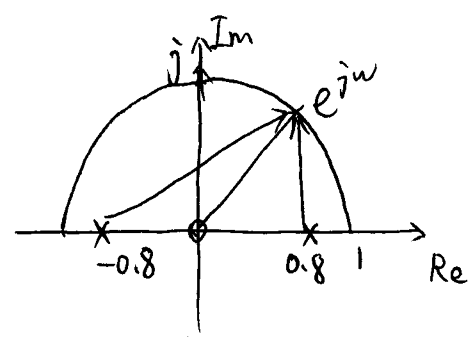
-- $a_1 = 0$, All pass filter; $y[n] = x[n]$

-- $-1 < a_1 < 0$ HPF

-- $|a_1| \geq 1$ non-stable $\sum_{n=0}^{\infty} |h[n]| = \sum_{n=0}^{\infty} |a_1|^n$ diverges.

A second-order IIR filter:





$y[n] = x[n] + 0.64\, y[n-2]$

$Y(z) = X(z) + 0.64\, z^{-2} Y(z)$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - 0.64 z^{-2}} = \frac{z^2}{z^2 - 0.64}$$

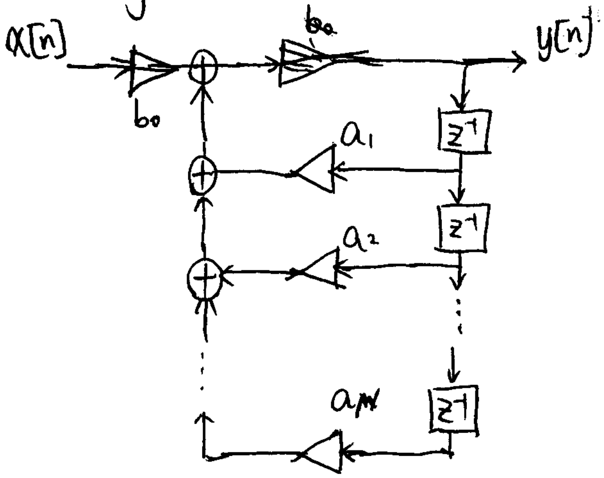
$$= \frac{z^2}{(z+0.8)(z-0.8)}$$

Zeros: $z=0$ twice

poles: $z = 0.8$, $z = -0.8$

$$|H(e^{jw})| = \frac{1}{|e^{jw} + 0.8|\,|e^{jw} - 0.8|}$$

A general LTI, causal, all-pole, IIR filter



$$y[n] = b_0 x[n] + \sum_{l=1}^{N} a_l \, y[n-l]$$

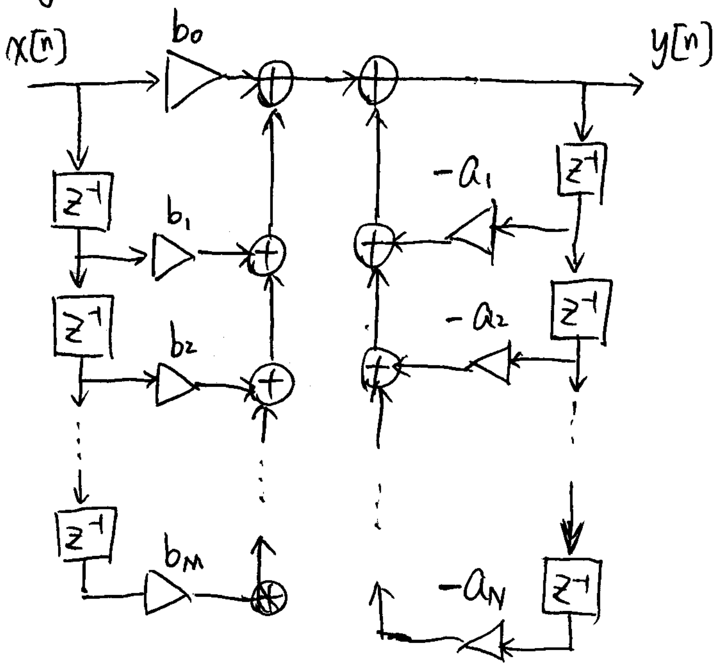
$$Y(z) = b_0 X(z) + \sum_{l=1}^{N} a_l z^{-l} Y(z)$$

$$H(z) = \frac{b_0}{1 - \sum_{l=1}^{N} a_l z^{-l}}$$

N ~~M~~ zeros, all at $z=0$

N ~~M~~ poles must be all within unit circle otherwise nonstable.

A general LTI, causal, IIR filter



$$y[n] = \sum_{k=0}^{M} b_k x[n-k] - \sum_{l=1}^{M} a_l \, y[n-l]$$

$$Y(z) = \sum_{k=0}^{M} b_k X(z) z^{-k} - \sum_{l=1}^{M} a_l z^{-l} Y(z)$$

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{l=1}^{N} a_l z^{-l}}$$

M nonzero zeros

N --- poles

Phase response: $H(e^{j\omega}) = |H(e^{j\omega})|\, e^{j\angle H(e^{j\omega})}$

$$\angle H(e^{j\omega}) = \tan^{-1}\frac{\operatorname{Im} H(e^{j\omega})}{\operatorname{Re} H(e^{j\omega})}$$

Look at the simplest FIR filter again

$$y[n] = x[n] + b_1 x[n-1]$$

$$Y(z) = 1 + b_1 z^{-1}$$

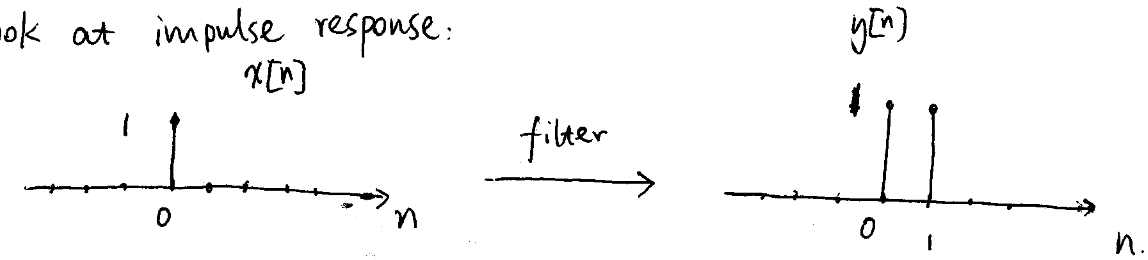
$$H(e^{j\omega}) = 1 + b_1 e^{-j\omega} = 1 + b_1\cos\omega - j b_1 \sin\omega$$

$$\therefore \angle H(e^{j\omega}) = \tan^{-1}\frac{-b_1\sin\omega}{1+b_1\cos\omega}$$

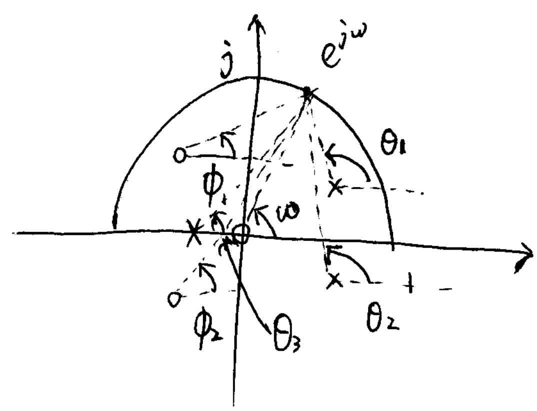
If $b_1 = 1$, $\tan^{-1}\frac{-\sin\omega}{1+\cos\omega} = \tan^{-1}\frac{-2\sin\frac{\omega}{2}\cos\frac{\omega}{2}}{2\cos^2\frac{\omega}{2}} = \tan^{-1}\left(-\tan\frac{\omega}{2}\right) = -\frac{\omega}{2}$

i.e, half-sample delay.

Look at impulse response:



$$H(z) = g\, z^{(N-M)} \frac{\prod_{l=1}^{M}(z-c_l)}{\prod_{k=1}^{N}(z-d_k)}$$



$$H(e^{j\omega}) = g\, e^{j(N-M)\omega} \frac{\prod_{l=1}^{M} A_l e^{j\phi_l}}{\prod_{k=1}^{N} B_k e^{j\theta_k}}$$

$$\therefore \angle H(e^{j\omega}) = \sum_{l=1}^{M}\phi_l - \sum_{k=1}^{N}\theta_k + (N-M)\omega$$