$$P_t(f) \approx \sum_z P(f|z) \cdot P_t(z)$$

The probability of one sound quanta : $P_t(f)$

Suppose sound quanta are ~~i.i.d.~~ independent to each other, we have $\sum_{f,t} V_{ft}$ Soundquan

Data likelihood, i.e. the probability of all sound quanta :

$$\prod_t \prod_f P_t(f)^{V_{ft}}, \quad \text{Log-likelihood} \quad \sum_t \sum_f V_{ft} \log P_t(f)$$

ML: ~~max~~ Maximize the data likelihood by adjusting the model parameters, i.e. $P(f|z)$ and $P_t(z)$.

For each sound quanta at time t, it has two random variables, f, and z

To completely describe a ~~sound~~ quanta, we should use $P_t(f,z)$.

EM algorithm.

Complete data likelihood $\prod_t \prod_f P_t(f,z)^{V_{ft}}, \quad \sum_t \sum_f V_{ft} \log P_t(f,z)$

M step: maximize the expected complete ~~log~~ data log likelihood.

$$\begin{aligned}
L &= E_{P_t(z|f)}\left[\sum_t \sum_f V_{ft} \log P_t(f,z)\right] \\
&= \sum_t \sum_f V_{ft}\, E_{P_t(z|f)}\left[\log P_t(f,z)\right] \\
&= \sum_t \sum_f V_{ft}\, E_{P_t(z|f)}\left[\log P(f|z) + \log P_t(z)\right] \\
&= \sum_t \sum_f V_{ft}\, E_{P_t(z|f)}\left[\log P(f|z)\right] + \sum_t \sum_f V_{ft}\, E_{P_t(z|f)}\left[\log P_t(z)\right] \\
&= \sum_t \sum_f V_{ft} \sum_z P_t(z|f) \log P(f|z) + \sum_t \sum_f V_{ft} \sum_z P_t(z|f) \log P_t(z)
\end{aligned}$$

s.t. $\sum_f P(f|z) = 1$ for all z.

$\sum_z P_t(z) = 1$ for all t

We can enforce these constraints by using Lagrange multipliers.

$$Q = L + \sum_z \rho_z \left(1 - \sum_f P(f|z)\right) + \sum_t \tau_t \left[1 - \sum_z P_t(z)\right]$$

$$\max_{P(f|z),\ P_t(z)} Q$$

Take derivatives:

for a particular f and z:

$$\frac{\partial Q}{\partial P(f|z)} = \sum_t V_{ft} P_t(z|f) \frac{1}{P(f|z)} - \rho_z = 0$$

$$\Rightarrow \sum_t V_{ft} P_t(z|f) = P(f|z)\, \rho_z$$

$$\Rightarrow \sum_f \sum_t V_{ft} P_t(z|f) = \sum_f P(f|z)\, \rho_z = \rho_z$$

$$\Rightarrow \boxed{P(f|z) = \frac{\sum_t V_{ft} P_t(z|f)}{\rho_z} = \frac{\sum_t V_{ft} P_t(z|f)}{\sum_f \sum_t V_{ft} P_t(z|f)}}$$

for a particular z and t:

$$\frac{\partial Q}{\partial P_t(z)} = \sum_f V_{ft} P_t(z|f) \frac{1}{P_t(z)} - \tau_t = 0$$

$$\Rightarrow \sum_f V_{ft} P_t(z|f) = P_t(z)\, \tau_t$$

$$\Rightarrow \sum_z \sum_f V_{ft} P_t(z|f) = \sum_z P_t(z)\, \tau_t = \tau_t$$

$$\Rightarrow \boxed{P_t(z) = \frac{\sum_f V_{ft} P_t(z|f)}{\tau_t} = \frac{\sum_f V_{ft} P_t(z|f)}{\sum_z \sum_f V_{ft} P_t(z|f)}}$$

E step: $P_t(z|f) = \dfrac{P_t(z,f)}{P_t(f)} = \dfrac{P(f|z)\,P_t(z)}{\sum_z P(f|z)\,P_t(z)}$

Compare with NMF-KL multiplicative update:

$$W_{ia} \leftarrow W_{ia} \frac{\sum_\mu H_{a\mu} V_{i\mu} / (WH)_{i\mu}}{\sum_\mu H_{a\nu}}$$

$$H_{a\mu} \leftarrow H_{a\mu} \frac{\sum_i W_{ia} V_{i\mu} / ((WH)_{i\mu}}{\sum_k W_{ka}}$$

$W_{ia} \sim P(f|z)$ $\quad$ $i \sim f$, $a \sim z$.

$H_{a\mu} \sim P_t(z)$ $\quad$ $a \sim z$, $\mu \sim t$.

~~E step:~~