

Continuous - connection, opposite of discrete.
all gaps filled. distinction b/w real numbers
(continuous) vs natural numbers (discrete).

Discrete things are countable; continuous materials
can always be subdivided.

Approximating matter as continuous allows more &
easier solutions of the physics. But the approx.
breaks down at small length scales. How small?



The average value of some generic
physical property f is our interest.

Consider density:
$$\rho = \frac{\sum_{i=1}^N m_i}{V}$$
mass of molecule i
scales $\sim N$

The continuum approximation is accurate if fluctuations
about the mean $\ll$ the mean.

From statistics, fluctuations in count $\delta N \sim \sqrt{N}$

$$\text{So } \frac{\text{fluctuation}}{\text{mean}} \sim \frac{\sqrt{N}}{N} = \frac{1}{\sqrt{N}}$$

Suppose a 10% error is OK for us. Need $N = 100$
molecules.



$$100^{1/3} = 4.64$$

Roughly, a $(1 \mu\text{m})^3$ volume contains 10^{10} molecules for

common liquids + solids or 10^7 for common gases (STP).
Often, continuum approx is accurate to a nm scale.

Tensors ch. 2 Lai

Tensors are mathematical quantities that respond to rotations and inversions of coordinate systems in a way that preserves their physical direction(s).

Components of a tensor may change during rotation/inversion in order to maintain physical direction.

Scalars are tensors of "order" 0. (AKA "rank" 0)

Vectors are tensors of order 1, having magnitude and direction, or equivalently, a component for each coordinate direction.

e.g. $\vec{X} = \hat{e}_1 x_1 + \hat{e}_2 x_2 + \hat{e}_3 x_3$

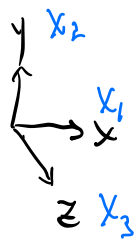
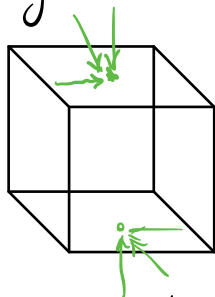
w/ $\hat{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\hat{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\hat{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ in matrix notation

or more concisely, we can write x_i to mean the i^{th} component (with $i \in \{1, 2, 3\}$). "indicial notation"

A scalar is a tensor of order 0. A vector is a tensor of order 1, having a component for each coordinate direction. An order-2 tensor has a component for each pair of coordinate directions. An order-3 tensor has a component for each triple of coordinate directions. And so on!

Example: Stress tensor. $\text{Stress} = \frac{\text{force}}{\text{area}}$

imagine a small cube of material ("material element")



To which side is stress applied? Normal to x or y or z .

In what direction is the stress applied? (x, y, z)

An order-2 tensor has 9 components, eg

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$$

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

A vector can be written $\vec{x}$. Any component can be written x_i or x_j or x_k ...

An order-2 tensor is often written boldface ($A, B, \dots$).

Any component of an order-2 tensor can

be written σ_{ij} or A_{ijk} or...

$$A_{ij} = \hat{e}_i \cdot A \hat{e}_j$$

Subscripts are (row, column)

Einstein Summation Convention

Whenever an index repeats within the same term, summation is implied.

$$\text{eg. } a_i b_i = \sum_{i=1}^3 a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3 = \vec{a} \cdot \vec{b}$$

Repeated indices are like dummy vars. for integration: name doesn't matter. $a_i b_i = a_j b_j = a_k b_k$ etc.

On the other hand, "free indices" must be homogeneous:

$$a_i = b_i + c_i \quad \text{valid}$$

$$a_j = b_i + c_k \quad \text{invalid!}$$

$$a_{ij} \mu_j = \lambda \mu_i \quad \text{valid!} \quad j \text{ repeated, } i \text{ free}$$

↑ order 2 ↑ order 1 ↑ order 0

$a_{ii} b_{ii}$ not valid! No index is allowed to appear more than twice in any term.

Useful for error checking!

Order of factors doesn't matter: $A_{ij} B_{jk} = B_{jk} A_{ij}$
index name dictates summing procedures.

Different from matrix notation! $AB \neq BA$

$$A_{ij} B_{jk} = \sum_{j=1}^3 A_{ij} B_{jk}$$

Matrix mult. if adjacent indices match.

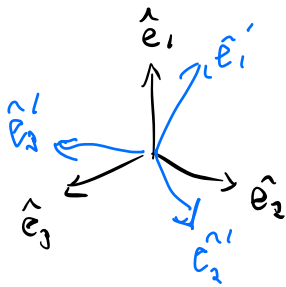
$$AB = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix} \begin{bmatrix} B_{11} \\ B_{21} \\ B_{31} \end{bmatrix} \begin{bmatrix} B_{13} \\ B_{23} \\ B_{33} \end{bmatrix}$$

i	k	$A_{ij} B_{jk}$
1	1	$A_{11} B_{11} + A_{12} B_{21} + A_{13} B_{31}$
1	2	
1	3	
2	1	
2	2	
2	3	$A_{21} B_{13} + A_{22} B_{23} + A_{23} B_{33}$ ✗
3	1	
3	2	
3	3	

What's $A_{ij} B_{kj}$? $= A_{ij} [B_{jk}]^T = A B^T$

Formal Definition of a tensor

First, order 1 tensor. Coord. sys. O : $\vec{x} = x_1 \hat{e}_1 + x_2 \hat{e}_2 + x_3 \hat{e}_3$
 O' : $\vec{x}' = x'_1 \hat{e}'_1 + x'_2 \hat{e}'_2 + x'_3 \hat{e}'_3$



Now $\vec{x} \cdot \hat{e}'_1 = x_1 \hat{e}_1 \cdot \hat{e}'_1 + x_2 \hat{e}_2 \cdot \hat{e}'_1 + x_3 \hat{e}_3 \cdot \hat{e}'_1$

abbreviation: $\hat{e}_i \cdot \hat{e}'_j = C_{ij} = \cos \theta_{ij}$

Component! $x'_1 = \vec{x} \cdot \hat{e}'_1 = x_1 C_{11} + x_2 C_{21} + x_3 C_{31}$

↖ angle between $\hat{e}_1$ & $\hat{e}'_1$

Generally, $x'_j = x_1 C_{1j} + x_2 C_{2j} + x_3 C_{3j}$

$x'_j = x_i C_{ij} \longleftrightarrow \vec{x}' = C \vec{x}$

↑ transformation

C transforms components in O to components in O' matrix

Any 3-element ^{component} quantity x_i for which $x'_j = x_i C_{ij}$

is an order-1 tensor. Likewise, any quantity that has a component for every pair of coordinate directions and transforms via $A'_{mn} = C_{in} C_{jm} A_{ij}$ is an order-2 tensor.

$\uparrow$ subscript $\uparrow$ order matches $\uparrow$

order 3: $A'_{lmn} = C_{il} C_{jm} C_{kn} A_{ijk}$

etc.

Continuum approx is accurate when $\frac{\text{deviation}}{\text{mean}} \ll 1$.

$$\frac{\text{deviation}}{\text{mean}} = \frac{\sqrt{\frac{1}{N} \sum_{i=1}^N (x_i - \bar{x})^2}}{\frac{1}{N} \sum_{i=1}^N x_i} \approx \frac{\sqrt{\frac{1}{N}}}{\frac{1}{N}} = \sqrt{N}$$

$\uparrow$ mean $\uparrow$ some of x_i $\uparrow$ molecule count $\uparrow$ molecule label

More tensor multiplication

$$A_{ij} b_j$$

↑
1 free index: 3 components
sum over j

$$a_i b_j$$

2 free indices
9 components
no summing

$$A_{ij} B_{ji}$$

0 free indices
1 component
sum over i & j

Go slow & follow the indices!
Analogy to matrices can
build intuition.

$$A_{ij} B_{jk} \Leftrightarrow A B$$

$$A_{ij} B_{kj} \Leftrightarrow A B^T$$

$$A_{ji} B_{jk} \Leftrightarrow A^T B$$

$$A_{ij} B_{ij} = \sum_{i=1}^3 \sum_{j=1}^3 A_{ij} B_{ij} \Leftrightarrow A:B$$

0 free indices
1 component
scalar product

$$= A_{11} B_{11} + A_{12} B_{12} + A_{13} B_{13} + \dots$$

$$A_{ij} B_{kl}$$

0 repeated indices
4 free indices
no summation
81 components
"outer product"

$$A \otimes B$$

$$a_i b_j$$

outer product
2 free indices
 $\vec{a} \otimes \vec{b}$

Cross product: $\vec{c} = \vec{a} \times \vec{b} \Leftrightarrow c_i = \epsilon_{ijk} a_j b_k$

where $\epsilon_{ijk} = \begin{cases} 1 & \text{if } \{i,j,k\} = \{1,2,3\} \text{ or } \{2,3,1\} \text{ or } \{3,1,2\} \\ -1 & \text{if } \{i,j,k\} = \{3,2,1\} \text{ or } \{2,1,3\} \text{ or } \{1,3,2\} \\ 0 & \text{otherwise (repeated index)} \end{cases}$

"permutation symbol"
"Levi-Civita"

i	j	k	$\epsilon_{ijk} a_j b_k$
1	1	1	0
	2	2	0
	3	3	0
1	2	3	+1 $a_2 b_3$
1	3	2	-1 $a_3 b_2$
2	1	3	-1 $a_1 b_3$
2	3	1	+1 $a_3 b_1$
3	1	2	+1 $a_1 b_2$
3	2	1	-1 $a_2 b_1$

$$C_1 = a_2 b_3 - a_3 b_2$$

$$C_2 = a_3 b_1 - a_1 b_3$$

$$C_3 = a_1 b_2 - a_2 b_1$$

Altogether, in vector notation

$$\vec{C} = (a_2 b_3 - a_3 b_2) \hat{e}_1$$

$$+ (a_3 b_1 - a_1 b_3) \hat{e}_2$$

$$+ (a_1 b_2 - a_2 b_1) \hat{e}_3$$

cross product!

Rotating the indices doesn't change the value of

$$\epsilon_{ijk} : \epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij} \quad \epsilon_{ijk} = \epsilon_{ikj} = \epsilon_{kji}$$

Exchanging 2 indices negates $\epsilon_{ijk} : \epsilon_{ijk} = -\epsilon_{jik}$ etc.

ϵ_{ijk} is a pseudo-tensor: it maintains its direction when coordinates are rotated but flips when coordinates are negated ("parity inversion").

Has handedness! Same is true for for qtps calc'd

w/ ϵ_{ijk} , eg ang. mom, torque, vorticity, helicity.

ϵ_{ijk} is the only order-3 isotropic tensor.

Another useful object: Kronecker delta

$$\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases} \longleftrightarrow \underline{1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

δ_{ij} operates by replacing i with j (or j with i), eg

$$\delta_{ij} u_j = \delta_{i1} u_1 + \delta_{i2} u_2 + \delta_{i3} u_3 = \begin{cases} u_1 & \text{if } i=1 \\ u_2 & \text{if } i=2 \\ u_3 & \text{if } i=3 \end{cases} \\ = u_i$$

$$\delta_{ij} A_{jk} = A_{ik} \quad \delta_{ij} A_{ij} = A_{ii} = A_{jj}$$

δ_{ij} is the only isotropic order-2 tensor.

Eigenvalues + Eigenvectors

If $A_{ij} a_j = \lambda a_i$ then λ is said to be an eigenvalue of A_{ij} (or A) and a_i (or $\vec{a}$) is said to be an eigenvector of A_{ij} (or A).

If A_{ij} is a transformation tensor, what does it do geometrically? $\rightarrow$ rotate + scale.

$A_{ij} \vec{a}_i$

Rotation about the $\vec{a}$ axis,

scaling by λ .

If you ^{know} A_{ij} , how to calc λ & a_i ?

$$A_{ij} a_j = \lambda a_i$$

$$A_{ij} a_j = \lambda \delta_{ij} a_j$$

$$(A_{ij} - \lambda \delta_{ij}) a_j = 0 \quad (1)$$

If we can multiply by the inverse of $(\sim)$,
 $a_j = 0$ is only solution (trivial case). Seek case
where $A_{ij} - \lambda \delta_{ij}$ has no inverse ("singular"). True if
 $\det(A_{ij} - \lambda \delta_{ij}) = 0$.

For 3D Cartesian, $\det \begin{pmatrix} A_{11} - \lambda & A_{12} & A_{13} \\ A_{21} & A_{22} - \lambda & A_{23} \\ A_{31} & A_{32} & A_{33} - \lambda \end{pmatrix} = 0 \quad (2)$

"characteristic eq'n"

Algebra gives 3 solutions for λ (eigenvalues). Use each
in Eq. (1) to get a corresponding eigenvector a_j .

Can write (2) as $-\text{III} + \text{II}\lambda - \text{I}\lambda^2 + \lambda^3 = 0$ where

$$\text{I} \equiv \delta_{ji} A_{ij} = A_{ii} = A_{11} + A_{22} + A_{33} \quad \begin{array}{l} \text{sum of diagonal components} \\ \text{"trace"} \quad \text{tr } A \end{array}$$

$$\text{II} = \frac{1}{2} \left((\delta_{ji} A_{ij})^2 - \delta_{ij} A_{im} A_{mj} \right)$$

$$\text{II} = \frac{1}{2} \left((\text{tr } A)^2 - \text{tr}(A^2) \right)$$

$$\text{III} \equiv \det(A_{ij}) \quad \det A$$

Together, I, II, & III are the "principal scalar invariants"

of A_{ij} . Values don't change with coordinate changes.

Cayley-Hamilton theorem: A itself satisfies its characteristic eq'n: $-\text{III} + \text{II}A - \text{I}A^2 + A^3 = 0$

Spectral theorem: If λ_i 's are eigenvalues of A
then $A = \sum_i \lambda_i \hat{e}_i \otimes \hat{e}_i$

Symmetric and Antisymmetric tensors

If $A_{ij} = A_{ji}$ (or equivalently, $A = A^T$) then A_{ij} is said to be "symmetric" with respect to i & j .

If $A_{ij} = -A_{ji}$ ($A = -A^T$) then A_{ij} is said to be "antisymmetric" with respect to i & j .

A_{ijk} might be symmetric or antisymmetric or neither with respect to i & j , j & k , k & i , all independently.

Any tensor A can be written as a unique sum of symmetric & antisymmetric parts:

$$A = A^S + A^A = \underbrace{\frac{1}{2}(A + A^T)}_{A^S} + \underbrace{\frac{1}{2}(A - A^T)}_{A^A}$$

Q: $\text{tr}(A^A)$? = 0 b/c $0 = -0$!

If A is order 2, how many components are in $A^S + A^A$? $\rightarrow$ 9 each. How many are independent?

9 total: A^A has 0 on diagonal, 3 independent.

A^S has 6 independent.

Any antisymmetric order-2 tensor A^A has a "dual vector" $\vec{k}^A$ such that $A^A \vec{a} = \vec{k}^A \times \vec{a}$ for any $\vec{a}$.

$$\vec{T}^A = \begin{bmatrix} A_{32}^A \\ A_{13}^A \\ A_{21}^A \end{bmatrix}$$

"Hodge dual" generalizes this idea to tensors of other order.

Orthogonal tensors

An orthogonal order-2 tensor $\mathbb{Q}$ transforms vectors such that

1. length is preserved: $|\mathbb{Q}\vec{a}| = |\vec{a}|$
2. angles between vectors are preserved: if the angle between $\vec{a}$ and $\vec{b}$ is α then the angle between $\mathbb{Q}\vec{a}$ and $\mathbb{Q}\vec{b}$ is also α .

Together, $(\mathbb{Q}\vec{a}) \cdot (\mathbb{Q}\vec{b}) = \vec{a} \cdot \vec{b}$ for any $\vec{a}, \vec{b}$

One example: coordinate transformation tensor $\mathbb{Q}$
Also, $\mathbb{Q}^T \mathbb{Q} = \mathbb{1}$, so $\mathbb{Q}^T = \mathbb{Q}^{-1}$. Also $\det(\mathbb{Q}) = 1$.

Matrix inverse

If $AA^{-1} = A^{-1}A = \mathbb{1}$ then A^{-1} is said to be the inverse of A . One way to calculate A^{-1} from A :

$$\left[\begin{array}{c|c} A & \mathbb{1} \end{array} \right]$$

then make linear combinations of rows (row reduction) until the form is

$$\left[\begin{array}{c|c} \mathbb{1} & A^{-1} \end{array} \right]$$

Ex: $\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array} \right]$ Replace $r1$ w/ $-2r1 + r2$

A

$\left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 3 & 4 & 0 & 1 \end{array} \right]$ Replace $r2$ w/ $\frac{1}{4}r2 - \frac{3}{4}r1$

$\left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{array} \right]$

$A^{-1}!$

Tensor Calculus

Math to quantify spatial + temporal variation.

- Gradient. Consider $\vec{\nabla} T = \text{grad}(T)$ "del" "nabla"
 where $\vec{\nabla} = \hat{e}_1 \frac{\partial}{\partial x_1} + \hat{e}_2 \frac{\partial}{\partial x_2} + \hat{e}_3 \frac{\partial}{\partial x_3}$ (an operation)
 or indicial notation, $\frac{\partial T}{\partial x_i}$ or $\partial_i T$ scalar

Gradient of a scalar is order 1.

For any tensor A , (order of $\vec{\nabla} A$) = (order of A) + 1.

Ex: $T = T_0 e^{-\frac{1}{2L^2}(x_1^2 + x_2^2)}$ Calculate $\vec{\nabla} T$.

$$\vec{\nabla} T = \hat{e}_1 T_0 \left(-\frac{1}{2L^2} 2x_1 \right) e^{-\frac{1}{2L^2}(x_1^2 + x_2^2)} + \hat{e}_2 T_0 \left(-\frac{1}{2L^2} 2x_2 \right) e^{-\frac{1}{2L^2}(x_1^2 + x_2^2)} + \hat{e}_3 0$$

$$= -\frac{T_0}{L^2} e^{-\frac{1}{2L^2}(x_1^2 + x_2^2)} (x_1 \hat{e}_1 + x_2 \hat{e}_2)$$

If T is temperature, units of $\vec{\nabla} T$ are $\frac{\text{temperature}}{\text{length}}$.

Can apply gradient to higher-order tensors, eg

$\frac{\partial u_i}{\partial x_j}$ order 2. 9 components: Rate of change of each component of u_i in each coordinate direction

$$\begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \dots & \dots \end{bmatrix}$$

2. Divergence. Consider $\vec{\nabla} \cdot \vec{u} \leftrightarrow \frac{\partial}{\partial x_i} u_i = \frac{\partial u_i}{\partial x_i} = \partial_i u_i$
 or $\text{div}(u_i)$ or $\text{div}(\vec{u})$. Order 0: one lower than the order of $\vec{u}$. "contraction" - any tensor multiplication that reduces order. $\frac{\partial}{\partial x_i}$ has units $\frac{1}{\text{length}}$.

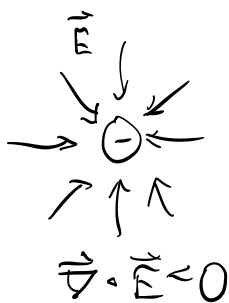
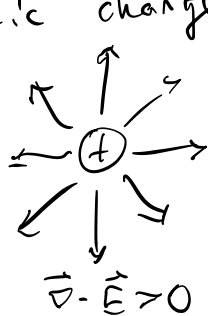
Ex: $\vec{V} = V_0 \left(\frac{x_1}{\sqrt{x_1^2 + x_2^2}} \hat{e}_1 + \frac{x_2}{\sqrt{x_1^2 + x_2^2}} \hat{e}_2 \right)$

Calculate $\vec{\nabla} \cdot \vec{V}$.

$$V_0 \frac{-x_2^2}{(x_1^2 + x_2^2)^{3/2}} + V_0 \frac{x_1^2}{(x_1^2 + x_2^2)^{3/2}} = \frac{V_0}{\sqrt{x_1^2 + x_2^2}} = \vec{\nabla} \cdot \vec{V}$$

charge density
↓
permittivity

Electric charges:



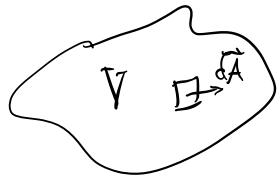
Maxwell: $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon}$

charge density
↓
permittivity

Can calculate divergence of higher-order tensors, eg

$$\frac{\partial \tau_{ij}}{\partial x_j}$$

2nd index matches



Gauss's law, AKA divergence theorem:

$$\oint_S \vec{u} \cdot d\vec{A} = \iiint_V \vec{\nabla} \cdot \vec{u} \, dV$$

$$\oint_S \vec{A} \cdot d\vec{A} = \iiint_V \text{div } \vec{A} \, dV$$

for any order

1. Gradients, 2. divergence

3. Curl. Consider $\vec{\omega} = \text{curl}(\vec{u}) = \vec{\nabla} \times \vec{u}$.

In indicial, $\omega_i = \epsilon_{ijk} \frac{\partial}{\partial x_j} u_k$ order 1. units $\frac{1}{\text{length}}$

Ex: calculate $\omega_i = \epsilon_{ijk} \frac{\partial}{\partial x_j} u_k$ if $\vec{u} = u_0 \left(\frac{-x_2}{\sqrt{x_1^2 + x_2^2}} \hat{e}_1 + \frac{x_1}{\sqrt{x_1^2 + x_2^2}} \hat{e}_2 \right)$.

$$\omega_1 = \epsilon_{123} \frac{\partial}{\partial x_2} u_3 + \epsilon_{132} \frac{\partial}{\partial x_3} u_2 = 0$$

$$\omega_2 = \epsilon_{231} \frac{\partial}{\partial x_3} u_1 + \epsilon_{213} \frac{\partial}{\partial x_1} u_3 = 0$$

$$\omega_3 = \epsilon_{312} \frac{\partial}{\partial x_1} u_2 + \epsilon_{321} \frac{\partial}{\partial x_2} u_1 = \frac{u_0}{\sqrt{x_1^2 + x_2^2}}$$

$$\vec{\omega} = \frac{u_0}{\sqrt{x_1^2 + x_2^2}} \hat{e}_3$$

Vorticity quantifies rotation!

Points along axis of rotation, by right-hand rule.

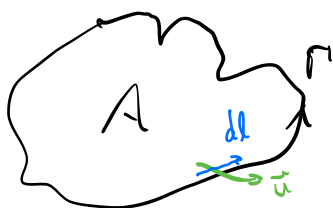
$\vec{\nabla} \times \vec{v}$ is "vorticity" if $\vec{v}$ is velocity.

Also, electromagnetics: $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ ← electric magnetic

$$\vec{\nabla} \times \vec{D} = \vec{J} + \frac{\partial \vec{P}}{\partial t}$$

$$\vec{\nabla} \times \vec{H} = \frac{\vec{J}}{\epsilon} + \frac{1}{\epsilon \mu} \frac{\partial \vec{B}}{\partial t}$$

Stokes theorem: $\oint_{\Gamma} \vec{u} \cdot d\vec{l} = \iint_A (\vec{\nabla} \times \vec{u}) \cdot d\vec{A}$ for any $\vec{u}$



component of $\vec{u}$ along path Γ , locally circulation

rotation locally

Now, indicial:

$$\oint_{\Gamma} \underbrace{u_i}_{\text{unit tangent}} t_i d\ell = \iint_A \underbrace{\epsilon_{kji} \frac{\partial}{\partial x_j} u_i}_{\text{unit normal}} n_k dA$$

Also applies to higher-order tensors:

$$\oint_{\Gamma} T_{ilm\dots} t_i d\ell = \iint_A \epsilon_{kji} \frac{\partial}{\partial x_j} T_{ilm\dots} n_k dA$$

4. Laplacian. Consider $\nabla^2 T$, or indicial, $\frac{\partial^2 T}{\partial x_i^2}$:

∇^2 does not change tensor order. units $\frac{1}{(\text{length})^2}$ curvature not slope

Tends to smooth things out.

Kinematics Lai ch. 3

Quantitative description of motion, not yet considering causes (force, momentum, energy). vs "dynamics"

We often think of individual particles (Lagrangian point of view). A particle's path is given by

$$\vec{x}(t) = x_1(t) \hat{e}_1 + x_2(t) \hat{e}_2 + x_3(t) \hat{e}_3$$

Which particle? Identify via an initial condition:

This particle is the one that had position $\vec{X} = \vec{X}$ at $t=0$.

$\vec{x}(t; \vec{X})$. Lagrangian view required for eg Newton's 2nd.

Alternate description (Eulerian): Forget particles altogether. Think of all quantities as fields, varying through

space + time. To specify uniquely, supply t + $\vec{x}$.
Emphasis on global coordinates.

Now think more about a particle. Total temporal rate of change of θ , for the particle initially at position $\vec{x}_i$ is $\left. \frac{D\theta}{Dt} \right|_{\vec{x}_i \text{ fixed}} = \left. \frac{\partial \theta}{\partial t} \right|_{\vec{x}_i \text{ fixed}} + \underbrace{\left(\frac{\partial \theta}{\partial x_1} \frac{\partial x_1}{\partial t} \right)}_{v_1} + \underbrace{\left(\frac{\partial \theta}{\partial x_2} \frac{\partial x_2}{\partial t} \right)}_{v_2} + \underbrace{\left(\frac{\partial \theta}{\partial x_3} \frac{\partial x_3}{\partial t} \right)}_{v_3}$

$$\left. \frac{D\theta}{Dt} \right|_{\vec{x}_i \text{ fixed}} = \left. \frac{\partial \theta}{\partial t} \right|_{\vec{x}_i \text{ fixed}} + (\vec{v} \cdot \vec{\nabla}) \theta$$

the operator: $\frac{D}{Dt} = \frac{\partial}{\partial t} + \underbrace{\vec{v} \cdot \vec{\nabla}}_{\substack{\text{convective} \\ \text{advective}}} \text{ or } \frac{D}{Dt} = \frac{\partial}{\partial t} + v_i \frac{\partial}{\partial x_i}$

is called the "material derivative". Expresses rate of change of q'tys experienced by a particle (Lagrangian) in terms of global coordinates + fields (Eulerian).

$$\vec{v} \cdot \vec{\nabla} = \underbrace{|\vec{v}|}_{\text{speed}} \underbrace{\hat{v} \cdot \vec{\nabla}}_{\text{component of gradient along } \hat{v} \text{ direction}}$$

$\vec{v}$ velocity

$$\vec{a} = \frac{D\vec{v}}{Dt} = \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{v}$$

$$a_i = \frac{\partial v_i}{\partial t} + v_j \frac{\partial}{\partial x_j} v_i$$

Q: meaning of $v_i \frac{\partial b_i}{\partial x_j} ? = v_i \frac{\partial}{\partial x_j} b_i = \vec{v} \cdot (\vec{\nabla} \vec{b})^T$

Contrast with
$$v_i \frac{\partial b_j}{\partial x_i} = \underbrace{v_i \frac{\partial}{\partial x_i}}_{\vec{v} \cdot \vec{\nabla}} b_j \quad (\vec{v} \cdot \vec{\nabla}) \vec{b}$$

Notation: eg $\frac{\partial u_i}{\partial x_j} = \partial_j u_i = u_{i,j}$. Or also $u_{i,jj}$

Material derivative: $\frac{D}{Dt} = \frac{\partial}{\partial t} + (\vec{u} \cdot \nabla)$
 temporal rate of change
 unsteady effects advective effects

Ex: A flow is given by $x_1 = \bar{x}_1 + \beta t$, $x_2 = \bar{x}_2 + \gamma t^2 \bar{x}_1$, $x_3 = \bar{x}_3$.
 Express the velocity in both Eulerian & Lagrangian terms

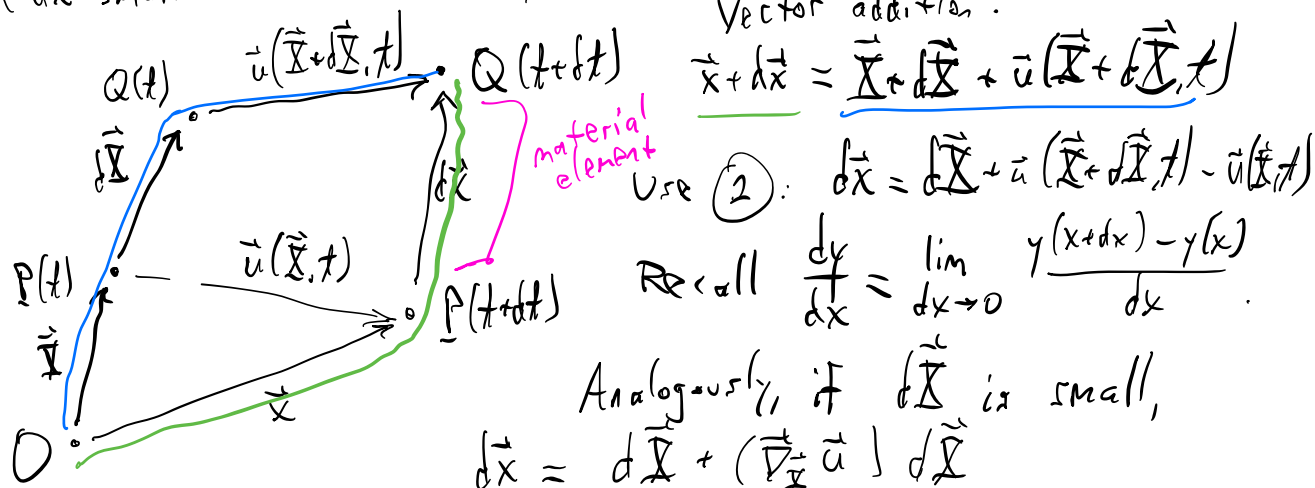
$\vec{v} = \frac{D\vec{x}}{Dt}$ $v_i = \frac{Dx_i}{Dt} = \frac{\partial x_i}{\partial t} + v_j \frac{\partial}{\partial x_j} x_i$ $\beta, \gamma, \bar{x}_i$ are const.

$v_1 = \beta$ $v_2 = 2\gamma t \bar{x}_1$ $v_3 = 0$ Lagrangian

Use (1): $v_1 = \beta$, $v_2 = 2\gamma(x_1 - \beta t)t$ $v_3 = 0$ Eulerian

Infinitesimal Deformations

Consider one particle P moving according to $\vec{x} = \vec{\bar{x}} + \vec{u}(\vec{\bar{x}}, t)$ (2).
 A nearby particle Q has initial location $\vec{\bar{x}} + d\vec{\bar{x}}$ and undergoes displacement $\vec{u}(\vec{\bar{x}} + d\vec{\bar{x}}, t)$ to location $\vec{\bar{x}} + d\vec{\bar{x}}$,
 ($d\vec{\bar{x}}$ small & time small).



displacement gradient

$$(\vec{\nabla}_{\vec{X}} \vec{u})_{ij} = \frac{\partial u_i}{\partial X_j}$$

$$d\vec{x} = (\mathbb{1} + \vec{\nabla}_{\vec{X}} \vec{u}) d\vec{X}$$

deformation gradient $\mathbb{F}$

$$F_{ij} = \delta_{ij} + \frac{\partial u_i}{\partial x_j}$$

If P & Q move in pure translation, $\vec{\nabla}_{\vec{X}} \vec{u} = 0$ and $\mathbb{F} = \mathbb{1}$
so $d\vec{x} = d\vec{X}$.

In general, how does distance b/w P & Q change?

Its square is $d\vec{x} \cdot d\vec{x} = d\vec{X}^T (\mathbb{1} + (\vec{\nabla}_{\vec{X}} \vec{u})^T) (\mathbb{1} + \vec{\nabla}_{\vec{X}} \vec{u}) d\vec{X}$ ←

$$\mathbb{F}^T \mathbb{F} = \mathbb{C}$$

unitless

right Cauchy-Green strain tensor

$$|d\vec{x}|^2 = d\vec{X}^T \mathbb{C} d\vec{X}$$

$\mathbb{C}$ quantifies the change in distance b/w P & Q . Expand:

$$\mathbb{C} = \mathbb{1} + \vec{\nabla} \vec{u} + (\vec{\nabla} \vec{u})^T + \cancel{(\vec{\nabla} \vec{u})^T \vec{\nabla} \vec{u}}$$

If $\vec{\nabla} \vec{u}$ small, can neglect ↑

Define "infinitesimal strain tensor" $\mathbb{E} = \frac{1}{2} (\vec{\nabla} \vec{u} + (\vec{\nabla} \vec{u})^T)$, the symmetric part of $\vec{\nabla} \vec{u}$. Also the "infinitesimal

rotation tensor" $\mathbb{Q} = \frac{1}{2} (\vec{\nabla} \vec{u} - (\vec{\nabla} \vec{u})^T)$, the antisymmetric part. Note $\vec{\nabla} \vec{u} = \mathbb{E} + \mathbb{Q}$ and $\mathbb{C} = \mathbb{1} + 2\mathbb{E}$.

To gain intuition for $\mathbb{E}$, consider 2 material elements:

$$\begin{aligned} d\vec{X}^{(1)} \xrightarrow{dt} d\vec{x}^{(1)} &= (\mathbb{1} + \vec{\nabla} \vec{u}) d\vec{X}^{(1)} \\ d\vec{X}^{(2)} \xrightarrow{dt} d\vec{x}^{(2)} &= (\mathbb{1} + \vec{\nabla} \vec{u}) d\vec{X}^{(2)} \end{aligned}$$

$$\text{So } d\vec{x}^{(1)} \cdot d\vec{x}^{(2)} = d\vec{X}^{(1)} \cdot d\vec{X}^{(2)} + 2 d\vec{X}^{(1)} \cdot \underline{E} d\vec{X}^{(2)} \leftarrow$$

Consider case 1: $d\vec{X}^{(1)} = d\vec{X}^{(2)} = dS \hat{e}_1$

$$\text{Then } d\vec{x}^{(1)} \cdot d\vec{x}^{(2)} = dS^2 + 2 dS^2 \underline{E}_{11} \underline{E} \hat{e}_1$$

Aside

$$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ d \\ g \end{bmatrix} = a$$

col 1 row 1

$$\underline{E}_{11} = \frac{ds^2 - dS^2}{2 dS^2} = \frac{(ds - dS)(ds + dS)}{2 dS^2} \approx \frac{(ds - dS)(2 dS)}{2 dS^2}$$

$$\underline{E}_{11} \approx \frac{ds}{dS} - 1$$

So if $ds = dS, \underline{E}_{11} = 0$
 $ds > dS, \underline{E}_{11} > 0$
 $ds < dS, \underline{E}_{11} < 0$

$\underline{E}_{11}$ quantifies elongation of material elements initially along $\hat{e}_1$.

$$\begin{array}{c} \underline{E}_{22} \\ \underline{E}_{33} \end{array} \quad \begin{array}{c} \hat{e}_2 \\ \hat{e}_3 \end{array}$$

Now case 2: $d\vec{X}^{(1)} = |d\vec{X}^{(1)}| \hat{e}_1$ and $d\vec{X}^{(2)} = |d\vec{X}^{(2)}| \hat{e}_2$.

$$\text{Then } d\vec{x}^{(1)} \cdot d\vec{x}^{(2)} = 0 + 2 |d\vec{X}^{(1)}| |d\vec{X}^{(2)}| \underline{E}_{12}$$

$|d\vec{x}^{(1)}| |d\vec{x}^{(2)}| \cos \theta$

$$\underline{E}_{12} = \frac{1}{2} \cos \theta \frac{|d\vec{x}^{(1)}|}{|d\vec{X}^{(1)}|} \frac{|d\vec{x}^{(2)}|}{|d\vec{X}^{(2)}|}$$

In the simple case where lengths don't change ($\underline{E}_{11} = \underline{E}_{22} = \underline{E}_{33} = 0$),

$\underline{E}_{12} = \frac{1}{2} \cos \theta$. So $\underline{E}_{12}$ quantifies the angle b/w material elements in the current configuration, if in the reference configuration, they were in the $\hat{e}_1 + \hat{e}_2$ directions.

$\underline{E}_{12}$ tell about angle change (AKA shear).

Analogous meanings for E_{23} & E_{31} .

Symmetric, so $E_{13} = E_{31}$, $E_{32} = E_{23}$, $E_{21} = E_{12}$.

The product of any symmetric & antisymmetric tensors is 0.

Deformation! Infinitesimal strain tensor $\mathbb{E} = \frac{1}{2}(\vec{\nabla}\vec{u} + (\vec{\nabla}\vec{u})^T)$

Diagonal components quantify elongation. Off-diagonal ^{$\frac{1}{2}\epsilon$} components quantify angle b/w material elements initially $\perp$ along coordinate directions.

What if $\mathbb{E}$ acts on $\vec{a}$ (its eigenvector)?

$$\mathbb{E}\vec{a} = \lambda_a \vec{a} \quad \text{also} \quad \mathbb{E}\vec{b} = \lambda_b \vec{b}, \quad \mathbb{E}\vec{c} = \lambda_c \vec{c} \quad (3 \text{ eigenvectors})$$

λ eigenvalue

$\vec{a}, \vec{b}, \vec{c}$ don't change direction. Angles b/w them go unchanged.

"principal directions" of the deformation encoded by $\mathbb{E}$.

λ 's are "principal strains" of $\mathbb{E}$. $\max(\lambda_a, \lambda_b, \lambda_c)$ is the largest strain in any direction.

$\min(\lambda_a, \lambda_b, \lambda_c)$ is smallest strain in any dir.

Rotation

The infinitesimal rotation tensor $\mathbb{Q} = \frac{1}{2}(\vec{\nabla}\vec{u} - (\vec{\nabla}\vec{u})^T)$.

$$d\vec{x} = (\mathbb{1} + \vec{\nabla}\vec{u})d\vec{X} = (\mathbb{1} + \mathbb{E} + \mathbb{Q})d\vec{X}$$

Consider the case where $d\vec{X}$ is an eigenvector of $\mathbb{E}$ with eigenvalue λ . Then $d\vec{x} = (1 + \lambda)d\vec{X} + \mathbb{Q}d\vec{X}$

Can show that Ω rotates the eigenvectors of $\mathbb{E}$:

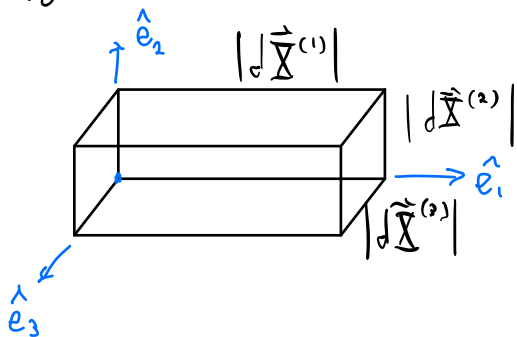
$$\Omega_{22} = \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3} \text{ is rotation angle about } \hat{e}_1$$

$$\Omega_{13} = \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1} \quad \text{---} \quad \hat{e}_2$$

$$\Omega_{21} = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \quad \text{---} \quad \hat{e}_3$$

Volume change

The fractional change of a volume is called its "dilatation".
To write it in terms of $\vec{\nabla} \vec{u}$, consider a block:



After deformation,

$$d\vec{X}^{(1)} \rightarrow d\vec{x}^{(1)} \text{ etc.}$$

$$\text{Recall that } |d\vec{x}^{(1)}| = (\mathbb{E}_{11} + 1) |d\vec{X}^{(1)}| \text{ etc.}$$

So the current volume relates to the reference volume

$$\text{via } \underbrace{|d\vec{x}^{(1)}| |d\vec{x}^{(2)}| |d\vec{x}^{(3)}|}_{dV} = (\mathbb{E}_{11} + 1)(\mathbb{E}_{22} + 1)(\mathbb{E}_{33} + 1) \underbrace{|d\vec{X}^{(1)}| |d\vec{X}^{(2)}| |d\vec{X}^{(3)}|}_{dV}$$

$$dV - dV = (\mathbb{E}_{11} + \mathbb{E}_{22} + \mathbb{E}_{33}) dV + \cancel{\mathbb{E}^2 \text{ terms}}_{\text{small}^2} + \cancel{\mathbb{E}^3 \text{ terms}}_{\text{small}^3}$$

$$\underbrace{\frac{dV - dV}{dV}}_{\text{dilatation}} = \mathbb{E}_{11} + \mathbb{E}_{22} + \mathbb{E}_{33} = \mathbb{E}_{ii} = \text{tr } \mathbb{E} = \frac{\partial u_i}{\partial x_i} = \vec{\nabla} \cdot \vec{u}$$

$$F_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

$$\begin{aligned} \text{tr } F &= \frac{1}{2} \left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_1}{\partial x_1} \right) \\ &+ \frac{1}{2} \left(\frac{\partial u_2}{\partial x_2} + \frac{\partial u_2}{\partial x_2} \right) \\ &+ \frac{1}{2} \left(\frac{\partial u_3}{\partial x_3} + \frac{\partial u_3}{\partial x_3} \right) \\ &= \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \frac{\partial u_i}{\partial x_i} \end{aligned}$$

dilatation: $\frac{dv - dV}{dV} = E_{ii} = \text{tr } E = \frac{\partial u_i}{\partial x_i} = \vec{\nabla} \cdot \vec{u}$

True for boxes, regardless of orientation, b/c trace is a scalar invariant. An "incompressible" deformation is one for which no dilatation occurs ($E_{ii} = 0$). Also:
 $\det F = 1 \quad (F = 1 + \vec{\nabla} \vec{u})$

Quantities for deformation rates

We define the "strain rate tensor" (AKA deformation rate) as $\mathbb{D} = \frac{1}{2} (\vec{\nabla}_{\vec{x}} \vec{v} + (\vec{\nabla}_{\vec{x}} \vec{v})^T)$

$\frac{1}{\text{time}}$
unitless

$\uparrow$
WRT
current config.

$\uparrow$
velocity

Analogous to $\mathbb{E}$! $\mathbb{D}$ is symmetric part of $\vec{\nabla}_{\vec{x}} \vec{v}$.

Gradients calc'd WRT current configuration b/c interested in instantaneous, current rates. Also define "rotation rate tensor" (AKA spin tensor): $\mathbb{W} = \frac{1}{2} (\vec{\nabla}_{\vec{x}} \vec{v} - (\vec{\nabla}_{\vec{x}} \vec{v})^T)$

Analogous to Ω ! $\vec{\nabla}_{\vec{x}} \vec{v} = \mathbb{D} + \mathbb{W}$.

By considering $d\vec{x}^{(1)}, d\vec{x}^{(1)}$ for a material element $d\vec{x}^{(1)}$ along a coordinate axis in the current config, can show $D_{ii} = \frac{1}{ds} \frac{D}{Dt} (ds)$

$\uparrow$ current length $|d\vec{x}^{(1)}|$

Rate of elongation along $\hat{e}_i$ dir in current config. Likewise for D_{22} & D_{33} .

D_{12} is half the rate of decrease of angle b/w 2 material elements currently along $\hat{e}_1$ & $\hat{e}_2$.
 "shear rate". Likewise for D_{23}, D_{31} .

tr $\mathbb{D} = D_{ii} = \vec{\nabla} \cdot \vec{v} = \frac{\partial v_i}{\partial x_i}$ is dilatation rate:

$$D_{ii} = \frac{1}{dV} \frac{D}{Dt} dV \quad \leftarrow \text{volume}$$

$\vec{W}$ has dual vector $\vec{\omega} = \vec{\nabla} \times \vec{v}$ "vorticity"
 $= - (W_{23} \hat{e}_1 + W_{31} \hat{e}_2 + W_{12} \hat{e}_3)$

Dynamics Lai ch. 4

Conservation of mass Lai sec. 3.15

Barring relativistic & nuclear effects, the mass of a particle remains unchanged.

$$0 = \frac{D}{Dt} (\rho dV) \quad \leftarrow \begin{array}{l} \text{density} \\ \text{volume} \end{array}$$

product rule: $\frac{D}{Dt} \rho dV + \rho \frac{D}{Dt} dV = 0$

Recall $\vec{\nabla} \cdot \vec{v} = \frac{1}{dV} \frac{D}{Dt} dV$, so

$$\frac{D}{Dt} \rho dV + \rho dV \vec{\nabla} \cdot \vec{v} = 0$$

$$\frac{D\rho}{Dt} + \rho \vec{\nabla} \cdot \vec{v} = 0 \quad \frac{\partial \rho}{\partial t} + v_i \frac{\partial \rho}{\partial x_i} + \rho \frac{\partial v_i}{\partial x_i} = 0$$

Cons. of mass! AKA "continuity"

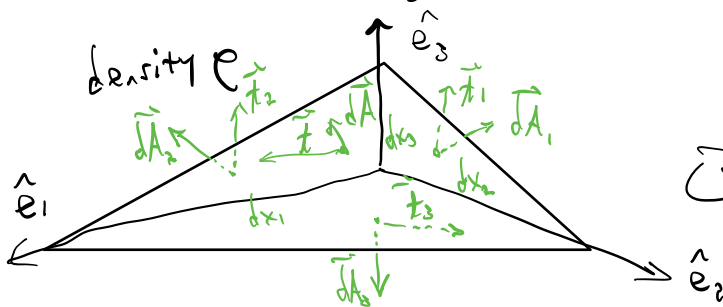
If $\rho = \text{constant}$, $\vec{\nabla} \cdot \vec{v} = 0$ or $\frac{\partial v_i}{\partial x_i} = 0$.

Stress tensor

Stress is force per unit area. Laws of mechanics depend on forces. Casting in terms of stresses removes size dependence. Need a tensor in order to express force couples, necessary for quantifying tension, compression, & shear. Stress tensor $\underline{\underline{T}}$ (AKA π or σ) is an order-2 (one component for every pair of coord. dirs) so we can specify orientation of a slice thru the mat'l and orientation of stress on that slice.

Consider the stress vector (AKA "traction") at some location on some surface in some continuous material. Depends on $\vec{x}$, t , and orientation: $\vec{t} = \vec{t}(\vec{x}, t, \hat{n})$ $\hat{n}$ unit normal

What if we change $\hat{n} \rightarrow -\hat{n}$?



Newton's 2ND says

$$\sum \vec{F} = m \vec{a}$$

$$\textcircled{1} \vec{t}_1 dA_1 + \vec{t}_2 dA_2 + \vec{t}_3 dA_3 + \vec{t} dA = \frac{1}{6} dx_1 dx_2 dx_3 \vec{a}$$

Geometry: volume $\frac{1}{6} dx_1 dx_2 dx_3$

If this material element is small, all dA 's are second-order small and $dx_1 dx_2 dx_3$ is third-order small.

RHS is negligible! Express in components:

$$\vec{dA} = dA \hat{n} = dA \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

$$dA_1 = -\hat{e}_1 \cdot \vec{dA} = -n_1 dA$$

$$dA_2 = -\hat{e}_2 \cdot \vec{dA} = -n_2 dA$$

$$dA_3 = -\hat{e}_3 \cdot \vec{dA} = -n_3 dA$$

into ①: $\vec{t} = \vec{t}_1 n_1 + \vec{t}_2 n_2 + \vec{t}_3 n_3$

$$\vec{t} = \underbrace{\begin{bmatrix} \vec{t}_1 \\ \vec{t}_2 \\ \vec{t}_3 \end{bmatrix}}_{\text{Cauchy stress tensor}} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

$$\vec{t} = \boldsymbol{\Pi} \hat{n}$$

$\boldsymbol{\Pi}$ "Cauchy stress tensor"

Σ T_{i2} gives stress on a slice $\perp \hat{e}_2$

T_{i1}

$\perp \hat{e}_1$

T_{i3}

$\perp \hat{e}_3$

T_{ij} is component of the stress in the $\hat{e}_i$ direction, on a slice $\perp$ to $\hat{e}_j$. Diagonal components specify normal stresses (tension/compression). Off-diagonal comp. specify shear stresses.

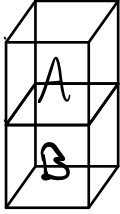
Stress tensor sign convention

1. $T_{ii} > 0$ signifies tensile stress along $\hat{e}_i$.
 $T_{ii} < 0$ compressive $\hat{e}_i$.

And likewise for T_{22} & T_{33} .

2. For off-diag terms, must decide whether, e.g.,
 $\begin{array}{c} \xrightarrow{T_{12}} \\ \xleftarrow{T_{21}} \end{array}$ or $\begin{array}{c} \xleftarrow{T_{12}} \\ \xrightarrow{T_{21}} \end{array}$ is positive. Consider 2

rectangular material elements in contact :



Last time

mass conservation $\frac{D\rho}{Dt} + \rho \vec{\nabla} \cdot \vec{v} = 0$

stress vector $\vec{t} = \mathbb{T} \hat{n}$
 $\uparrow$ unit normal
 $\uparrow$ stress tensor

Conservation of Angular Momentum

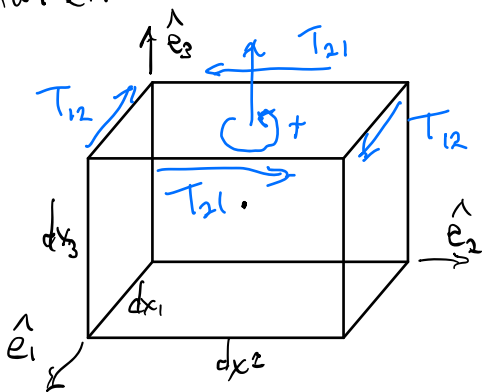
AKA moment of momentum.

$\sum \vec{\tau} = \mathbb{I} \vec{\alpha}$
 $\uparrow$ torque $\uparrow$ moment of inertia $\uparrow$ angular acceleration

Newton's 2nd law for rotation:

like $\sum \vec{F} = m \vec{a}$
 $\uparrow$ force $\uparrow$ mass $\uparrow$ accel

Plan: write in terms of $\mathbb{T}$, for a small, rectangular material element of uniform density ρ .



First consider the axis of rotation to be along $\hat{e}_3 \Rightarrow$ thru center. $\vec{\alpha} \rightarrow \alpha_3$

$\mathbb{I} \rightarrow \int_V \rho dx_1 dx_2 dx_3 (dx_1^2 + dx_2^2)$

Torques are caused by forces.

Write in terms of stresses. $\vec{t} = \mathbb{T} \hat{n}$, taking $\hat{n}$ to be outward normal.

$\hat{e}_1$ component of stress on right side is $\hat{e}_1 \cdot \mathbb{T} \hat{e}_2 = T_{12}$

$\hat{e}_1$ " " " " left " $\hat{e}_1 \cdot \mathbb{T}(-\hat{e}_1) = -T_{12}$
 $\hat{e}_2$ " " " " front " $\hat{e}_2 \cdot \mathbb{T} \hat{e}_1 = T_{21}$
 $\hat{e}_2$ " " " " back " $\hat{e}_2 \cdot \mathbb{T}(-\hat{e}_1) = -T_{21}$

so we get

$$\begin{aligned}
 & \underbrace{T_{21}}_{\text{stress}} \underbrace{dx_2 dx_3}_{\text{area of front}} \underbrace{\frac{dx_1}{2}}_{\text{moment arm}} + \underbrace{T_{21}}_{\text{back}} \underbrace{dx_2 dx_3}_{\text{area of back}} \underbrace{\frac{dx_1}{2}}_{\text{moment arm}} \\
 & - \underbrace{T_{12}}_{\text{right}} \underbrace{dx_1 dx_3}_{\text{area of right}} \underbrace{\frac{dx_2}{2}}_{\text{moment arm}} - \underbrace{T_{12}}_{\text{left}} \underbrace{dx_1 dx_3}_{\text{area of left}} \underbrace{\frac{dx_2}{2}}_{\text{moment arm}} \\
 & = \rho_{12} dx_1 dx_2 dx_3 (dx_1^2 + dx_2^2) \alpha_3
 \end{aligned}$$

Box is small, so RHS is second-order small + negligible.

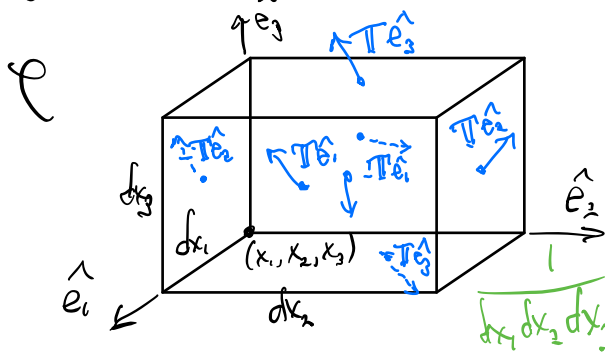
Altogether, $T_{12} = T_{21}$. Likewise find $T_{32} = T_{23}$ by calculating the $\hat{e}_1$ component and $T_{13} = T_{31}$ by calculating the $\hat{e}_2$ component.

$$T_{ij} = T_{ji} \quad \mathbb{T} = \mathbb{T}^T$$

Conservation of angular momentum is assured as long as $\mathbb{T}$ is symmetric.

Conservation of Linear Momentum

Newton's 2ND for translation: $\sum \vec{F} = m \vec{a}$. Again consider same material element:



surface forces and
body force $\rho \vec{b} dx_1 dx_2 dx_3$

$$\begin{aligned}
 & \underbrace{-\mathbb{T}(x_1, x_2, x_3) \hat{e}_1}_{\text{back}} dx_2 dx_3 + \underbrace{\mathbb{T}(x_1 + dx_1, x_2, x_3) dx_2 dx_3 \hat{e}_1}_{\text{front}} \\
 & \underbrace{-\mathbb{T}(x_1, x_2, x_3) \hat{e}_2}_{\text{left}} dx_1 dx_3 + \underbrace{\mathbb{T}(x_1, x_2 + dx_2, x_3) dx_1 dx_3 \hat{e}_2}_{\text{right}} \\
 & \underbrace{-\mathbb{T}(x_1, x_2, x_3) \hat{e}_3}_{\text{bottom}} dx_1 dx_2 + \underbrace{\mathbb{T}(x_1, x_2, x_3 + dx_3) dx_1 dx_2 \hat{e}_3}_{\text{top}}
 \end{aligned}$$

$$\begin{aligned}
& -T(x_1, x_2, x_3) \hat{e}_3 + T(x_1, x_2, x_3 + dx_3) \hat{e}_3 \\
& + \rho \vec{b} dx_1 dx_2 dx_3 = \rho dx_1 dx_2 dx_3 \vec{a} \} \\
& \frac{T(x_1 + dx_1, x_2, x_3) - T(x_1, x_2, x_3)}{dx_1} \hat{e}_1 + \frac{T(x_1, x_2 + dx_2, x_3) - T(x_1, x_2, x_3)}{dx_2} \hat{e}_2 \\
& + \frac{T(x_1, x_2, x_3 + dx_3) - T(x_1, x_2, x_3)}{dx_3} \hat{e}_3 + \rho \vec{b} = \rho \vec{a}
\end{aligned}$$

Now, material element is small.

$$\frac{\partial T}{\partial x_1} \hat{e}_1 + \frac{\partial T}{\partial x_2} \hat{e}_2 + \frac{\partial T}{\partial x_3} \hat{e}_3 + \rho \vec{b} = \rho \vec{a}$$

$$\text{div } \vec{T} + \rho \vec{b} = \rho \vec{a}$$

$$\frac{\partial T_{ij}}{\partial x_j} + \rho b_i = \rho a_i$$

Cauchy's equation of motion

Q: How much info req'd to calc. $\vec{a}$?

ρ , 3 components of $\vec{b}$, 6 independent components of $\vec{T}$
= 10 numbers

Ex: What if $T_{ij} = -p \delta_{ij}$ pressure in $\hat{a}_n$ (diagonal components)

A state of stress with no shear and isotropic normal stresses.

So Cauchy's eqn becomes $\frac{\partial}{\partial x_j} (-p \delta_{ij}) + \rho b_i = \rho a_i$

$$-\frac{\partial p}{\partial x_j} \delta_{ij} - p \frac{\partial \delta_{ij}}{\partial x_j} + \rho b_i = \rho a_i$$

$$-\frac{\partial A}{\partial x_i} + e\phi_i = e a_i$$

$$\vec{a} = -\frac{1}{e} \vec{\nabla} \rho + \vec{b}$$

Conservation of energy

AKA 1st law of thermo

$$\frac{D}{Dt} U + \frac{D}{Dt} KE = \dot{P} + \dot{Q}$$

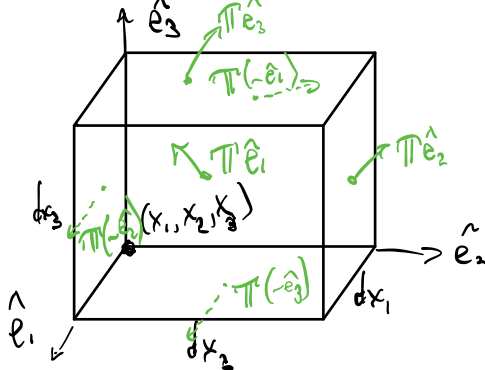
$\frac{D}{Dt}$ internal energy $\frac{D}{Dt}$ kinetic energy $\dot{P}$ rate of work done on system
 shape & volume change stress power $\dot{Q}$ rate of heat inflow

$$U = u \rho dV$$

u internal energy per unit mass $dV = dx_1 dx_2 dx_3$ volume

Mission: write in terms of $\mathbb{T}$ & $\mathbb{D}$.

stress power $\dot{P}$



$$\dot{P} = \rho dx_1 dx_2 dx_3 \vec{b} \cdot \vec{v}$$

velocity

$$\begin{aligned}
 & + \mathbb{T}(x_1 + \frac{dx_1}{2}, x_2 + \frac{dx_2}{2}, x_3 + \frac{dx_3}{2}) \hat{e}_1 \cdot \vec{v} dx_2 dx_3 \text{ front} \\
 & - \mathbb{T}(x_1, x_2 + \frac{dx_2}{2}, x_3 + \frac{dx_3}{2}) \hat{e}_1 \cdot \vec{v} dx_2 dx_3 \text{ back} \\
 & + \mathbb{T}(x_1 + \frac{dx_1}{2}, x_2 + \frac{dx_2}{2}, x_3 + \frac{dx_3}{2}) \hat{e}_2 \cdot \vec{v} dx_1 dx_3 \text{ right} \\
 & - \mathbb{T}(x_1 + \frac{dx_1}{2}, x_2, x_3 + \frac{dx_3}{2}) \hat{e}_2 \cdot \vec{v} dx_1 dx_3 \text{ left} \\
 & + \mathbb{T}(x_1 + \frac{dx_1}{2}, x_2 + \frac{dx_2}{2}, x_3 + \frac{dx_3}{2}) \hat{e}_3 \cdot \vec{v} dx_1 dx_2 \text{ top} \\
 & - \mathbb{T}(x_1 + \frac{dx_1}{2}, x_2 + \frac{dx_2}{2}, x_3) \hat{e}_3 \cdot \vec{v} dx_1 dx_2 \text{ bottom}
 \end{aligned}$$

$$\lim_{\substack{dx_1 \rightarrow 0 \\ dx_2 \rightarrow 0 \\ dx_3 \rightarrow 0}} \frac{\dot{P}}{dx_1 dx_2 dx_3} = \rho \vec{b} \cdot \vec{v} + \frac{\partial}{\partial x_1} (\mathbb{T} \hat{e}_1 \cdot \vec{v}) + \frac{\partial}{\partial x_2} (\mathbb{T} \hat{e}_2 \cdot \vec{v}) + \frac{\partial}{\partial x_3} (\mathbb{T} \hat{e}_3 \cdot \vec{v})$$

$$= \rho b_i v_i + \frac{\partial}{\partial x_j} (v_i T_{ij})$$

$$= \rho b_i v_i + v_i \frac{\partial T_{ij}}{\partial x_j} + T_{ij} \frac{\partial v_i}{\partial x_j}$$

$$= v_i \left(\frac{\partial T_{ij}}{\partial x_j} + \rho b_i \right)$$

$$\rho a_i = \rho \frac{Dv_i}{Dt}$$

from momentum eq.

$$\frac{1}{2} \left(T_{ij} \frac{\partial v_i}{\partial x_j} + T_{ij} \frac{\partial v_j}{\partial x_i} \right)$$

rename $i \rightarrow j, j \rightarrow i$

$$T_{ji} \frac{\partial v_j}{\partial x_i}$$

$$\frac{P}{dx_1 dx_2 dx_3} = \underbrace{\rho v_i \frac{Dv_i}{Dt}}_{\text{reverse product rule } \frac{1}{2} \frac{D}{Dt} (v_i v_i)} + T_{ij} D_{ij}$$

Π is symmetric!

$$T_{ij} \frac{\partial v_j}{\partial x_i}$$

$$T_{ij} \underbrace{\frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)}_{\Pi} D_{ij}$$

$$\frac{P}{dx_1 dx_2 dx_3} = \frac{D}{Dt} KE + T_{ij} D_{ij}$$

Rate of heat inflow Q

Can express as sum of inflows thru 6 sides. Each looks

like $-\hat{e}_1 \cdot \vec{q}$ $dx_2 dx_3$

↑
into the system

heat flux ($\frac{J}{s \cdot m^2}$)

Compare to mass flux,
momentum flux, entropy flux,

$$\begin{aligned} \sum Q = & -\hat{e}_1 \cdot \vec{q}(x_1+dx_1, x_2+\frac{dx_2}{2}, x_3+\frac{dx_3}{2}) dx_2 dx_3 & \text{front} \\ & + \hat{e}_1 \cdot \vec{q}(x_1, x_2+\frac{dx_2}{2}, x_3+\frac{dx_3}{2}) dx_2 dx_3 & \text{back} \\ & - \hat{e}_2 \cdot \vec{q}(x_1, x_2, x_3+\frac{dx_3}{2}) dx_1 dx_3 & \text{right} \\ & + \hat{e}_2 \cdot \vec{q}(x_1, x_2, x_3) dx_1 dx_3 & \text{left} \\ & - \hat{e}_3 \cdot \vec{q}(x_1, x_2, x_3) dx_1 dx_2 & \text{top} \\ & + \hat{e}_3 \cdot \vec{q}(x_1, x_2, x_3-dx_3) dx_1 dx_2 & \text{bottom} \end{aligned}$$

$$\lim_{dx \text{ small}} \frac{Q}{dx_1 dx_2 dx_3} = - \frac{\partial q_1}{\partial x_1} - \frac{\partial q_2}{\partial x_2} - \frac{\partial q_3}{\partial x_3} \quad \text{w/ } \vec{q} = q_1 \hat{e}_1 + q_2 \hat{e}_2 + q_3 \hat{e}_3$$

$$Q = -\vec{\nabla} \cdot \vec{q} dx_1 dx_2 dx_3$$

Altogether, $\rho \frac{Du}{Dt} + \frac{D}{Dt} KE = \frac{D}{Dt} KE + T_{ij} D_{ij} - \frac{\partial q_i}{\partial x_i}$

$$\rho \frac{Du}{Dt} = T_{ij} D_{ij} - \frac{\partial q_i}{\partial x_i} \quad \rho \frac{Du}{Dt} = \text{tr}(\Pi D) - \vec{\nabla} \cdot \vec{q}$$

$\Pi = D$

2nd law of thermo: Entropy

Entropy measures disorder. The entropy of a closed system never decreases. The entropy increase of a material element is never less than the entropy inflow across its boundaries. Define $\eta(\vec{x}, t)$ as the entropy per unit mass. Rate of incr. of entropy of mat'l element is $\frac{D}{Dt} (\eta dV) = \rho dV \frac{D\eta}{Dt}$

By definition, entropy flux is $\frac{\vec{q}}{\theta}$ ← temperature
so 2nd law requires $\rho dV \frac{D\eta}{Dt} \geq - \vec{\nabla} \cdot \frac{\vec{q}}{\theta} dV$

$$\rho \frac{D\eta}{Dt} \geq - \vec{\nabla} \cdot \frac{\vec{q}}{\theta}$$

Satisfaction guaranteed as long as material diffusion constants are not negative. (kinematic viscosity AKA momentum diffusivity, heat diffusivity, material diffusivity, magnetic diffusivity) $\frac{m^2}{s}$

Laws of Mechanics for Continuous Materials

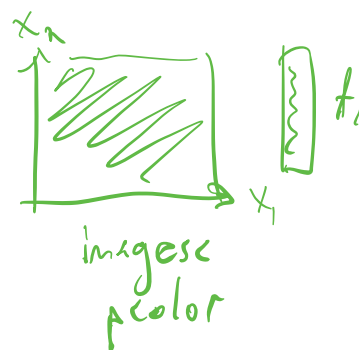
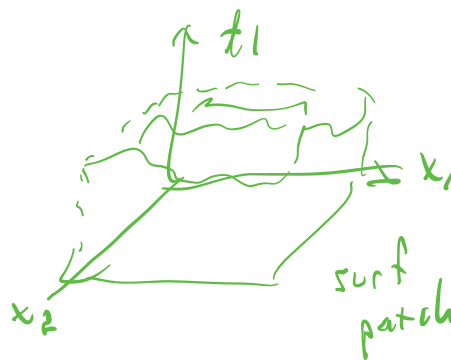
Mass: $\frac{D\rho}{Dt} + \rho \vec{\nabla} \cdot \vec{v} = 0$

Ang. Mom: $\mathbb{T} = \mathbb{T}^T$

Linear mom.: $\text{div } \mathbb{T} + \rho \vec{b} = \rho \vec{a}$

1st law: $\rho \frac{Du}{Dt} = \text{tr}(\mathbb{T} \mathbb{D}) - \vec{\nabla} \cdot \vec{q}$

2nd law: $\rho \frac{D\eta}{Dt} \geq - \vec{\nabla} \cdot \frac{\vec{q}}{\theta}$



"Equilibrium" means $\vec{a} = \vec{d} = 0$.

"Material surface" means $\frac{\partial}{\partial t} = 0$.

Elastic Solids ch. 4 Lai

Simplest def of solid: fixed shape. (Equivalently, molecules don't move past each other.) No deformation!
 $\vec{u} = 0$ Rigid body. An inflexible model!

Slightly better model: a linear relationship between stress and strain. 1D version: $F = -kx$ (Hooke)

Now, 3D!

$$T_{ij} = C_{ijkl} \epsilon_{kl}$$

C_{ijkl} elasticity tensor (Pa). 3⁴ = 81 components
 The most general linear relation between stress & strain allows any stress component T_{ij} to be a weighted sum of the strain components.

Cons. of ang mom. requires $T_{ij} = T_{ji}$, so
 $C_{ijkl} = C_{jikl}$. only 54 free components.

↑
force
= stress × area

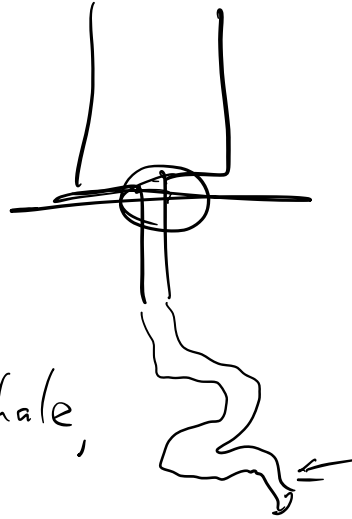
↑ elongation
= strain × initial length

Def: $E_{kl} = E_{lk}$, so $C_{ijkl} = C_{ijlk}$. Only 36 free components
Cons. of energy requires $C_{ijkl} = C_{klij}$, leaving just 21 components free.

Particular case: Isotropic

Isotropic solids: Al, polymers, glass, ceramics, steel, other metals, concrete

Anisotropic solids: wood, mica, shale, crystal, reinforced concrete, carbon fiber



A linear, elastic solid is a material for which stress varies linearly with strain. 1D: $F = -kx$.

3D: $T_{ij} = C_{ijkl} E_{kl}$

elasticity tensor. 81 components, 21 independent.

Special case: isotropic materials

Want C_{ijkl} to be isotropic. Recall that the only isotropic order-2 tensor is δ_{ij} , and ... order-3 is ϵ_{ijk} . Any isotropic order-4 tensor can be written

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \alpha \delta_{ik} \delta_{jl} + \beta \delta_{il} \delta_{jk}$$

$$T_{ij} = (\lambda \delta_{ij} \delta_{kl} + \alpha \delta_{ik} \delta_{jl} + \beta \delta_{il} \delta_{jk}) E_{kl}$$

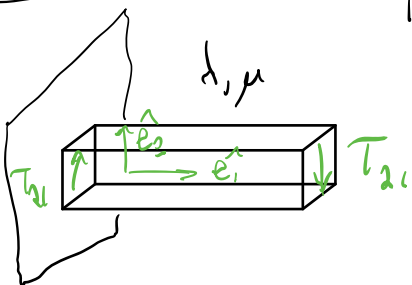
$$T_{ij} = \lambda \delta_{ij} E_{kk} + \alpha E_{ij} + \beta E_{ji} \quad E_{ji} = E_{ij} \text{ (symmetric)}$$

$$T_{ij} = \lambda \delta_{ij} E_{kk} + 2\mu E_{ij} \quad (1) \quad (2\mu = \alpha + \beta)$$

Can describe mechanics of an isotropic linear elastic solid w/ just 2 scalars, the "Lamé coefficients." units: Pa or equivalent.

Continue w/ thought experiments to gain intuition.

Shear test



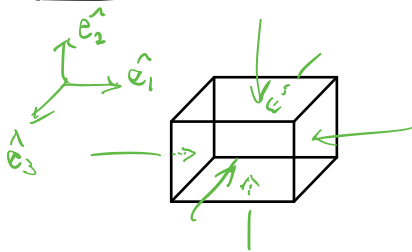
T_{21} known. All other $T_{ij} = 0$.

Use (1): $T_{21} = \lambda \delta_{21} E_{kk} + 2\mu E_{21}$

$$\mu = \frac{T_{21}}{2E_{21}} \leftarrow \begin{array}{l} \text{shear stress} \\ \text{angle change} \end{array}$$

μ is the stress per unit angle of flex,
called "shear modulus," often G .

Compression test



Apply uniform normal stress.

$$T_{ij} = -p \delta_{ij}$$

Measure dilatation: E_{kk} . Use (1):

$$\delta_{ij} \{ T_{ij} = \lambda \delta_{ij} E_{kk} + 2\mu E_{ij} \}$$

$$T_{ii} = \lambda \delta_{ii} E_{kk} + 2\mu E_{ii}$$

$$T_{ii} = (3\lambda + 2\mu) E_{kk}$$

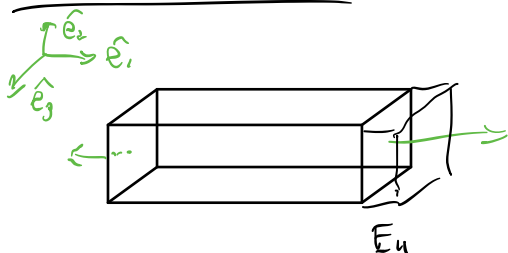
$$E_{kk} = \frac{T_{ii}}{3\lambda + 2\mu} = \frac{-3p}{3\lambda + 2\mu}$$

Bulk modulus is $\frac{\text{compressive stress}}{\text{dilatation}}$, $k = \frac{p}{\frac{3p}{3\lambda + 2\mu}} =$

$$k = \lambda + \frac{2}{3}\mu$$

A "modulus" is always $\frac{\text{stress}}{\text{strain}}$

Stretch test



All $T_{ij} = 0$ except T_{11} .

The Young's modulus is defined as

$$E_Y = \frac{\text{longitudinal stress}}{\text{longitudinal strain}} = \frac{T_{11}}{E_u}$$

use (1): $T_{ii} = \lambda \delta_{ii} E_{kk} + 2\mu E_{ii}$

use $E_{kk} = \frac{T_{ii}}{3\lambda + 2\mu}$ from above to get $T_{ii} = \frac{\lambda T_{ii}}{3\lambda + 2\mu} + 2\mu E_{ii}$

$T_{ii} + T_{jj} + T_{kk}$

Altogether, $E_Y = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}$

isotropic; can use either E_{22} or E_{33}

Poisson's ratio $\nu = - \frac{\text{lateral strain}}{\text{longitudinal strain}} = - \frac{E_{22}}{\frac{1}{2\mu}(T_{ii} - \lambda E_{kk} \delta_{ii})}$

unitless

$= - \frac{\frac{1}{2\mu}(T_{22} - \lambda E_{kk} \delta_{22})}{\frac{1}{2\mu}(T_{ii} - \lambda E_{kk} \delta_{ii})}$

$\nu = \frac{\lambda}{2(\lambda + \mu)}$

A linear, elastic, isotropic solid is fully characterized by any 2 of $\{\lambda, \mu, \nu, k, E_Y\}$. Others are redundant ... but convenient & easy to calc. algebraically.

Q: A material of known E_Y is incompressible. Determine $\{\nu, \mu, \lambda, k\}$.

$\nu = \frac{1}{2}$

$E_Y = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}$

$\nu = \frac{\lambda}{2(\lambda + \mu)} = \frac{1}{2}$

solve simult.
 $\mu = \frac{1}{3} E_Y$

use

$\frac{1}{2} = \frac{\lambda}{2(\lambda + \frac{1}{3} E_Y)}$

$\lambda = \frac{2}{3} \nu E_Y$

$\lambda = \infty$

$k = \lambda + \frac{2}{3} \mu = \infty$

No compression, regardless of compressive stress!

$\frac{1-2\nu}{1-2 \cdot \frac{1}{2}} = 0$

Solids! Linear, elastic, isotropic solid: $T_{ij} = \lambda E_{kk} \delta_{ij} + 2\mu E_{ij}$
 Bulk modulus, shear modulus, Young's modulus

↑
"constitutive relation"

A "constitutive relation" is a quantitative relation, for a particular material, linking stress to strain and/or strain rate. Can combine with equations of physics for generic continuum material.

Momentum, small deformation of elastic solid

$$T_{ij} = \lambda E_{kk} \delta_{ij} + 2\mu E_{ij} \quad \rho a_i = \frac{\partial T_{ij}}{\partial x_j} + \rho b_i$$

$$\frac{\partial T_{ij}}{\partial x_j} = \lambda \delta_{ij} \underbrace{\frac{\partial E_{kk}}{\partial x_j}}_{\frac{\partial E_{kk}}{\partial x_i}} + 2\mu \underbrace{\frac{\partial E_{ij}}{\partial x_j}}_{2\mu \frac{1}{2} \left(\frac{\partial}{\partial x_j} \frac{\partial u_i}{\partial X_j} + \frac{\partial}{\partial x_j} \frac{\partial u_j}{\partial X_i} \right)}$$

If deformation is small and slow,

$$\text{def: } x_i = X_i + u_i$$

$$\text{so } \frac{\partial}{\partial x_i} = \frac{\partial}{\partial X_i} + \cancel{\frac{\partial}{\partial u_i}}$$

$$\underbrace{\frac{\partial u_j}{\partial X_j}}_{E_{jj}}$$

$$a_i = \frac{Dv_i}{Dt} = \frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j}$$

$\uparrow$ small $\uparrow$ small² Neglect!

Likewise, for small & slow deformation, $v_i \approx \frac{\partial x_i}{\partial t}$
 and so $a_i \approx \frac{\partial^2 u_i}{\partial t^2}$

Altogether,

$$\rho \frac{\partial^2 u_i}{\partial t^2} = (\lambda + \mu) \frac{\partial E_{jj}}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j} + c^b_i$$

Navier equation of motion

Plane irrotational waves

Looks like

$$\frac{\partial^2 \phi}{\partial t^2} = c^2 \frac{\partial^2 \phi}{\partial x^2}$$

$\uparrow$ time $\uparrow$ position $\nwarrow$ scalar

which has solutions $\phi = \phi_c \cos k(x-ct) + \phi_s \sin k(x-ct)$

$\uparrow$ constant $\uparrow$ wave number $\uparrow$ wave speed $\uparrow$ constant

A wave number has units radians/m or similar & quantifies spatial frequency of oscillation.

Could also write $k(x-ct)$ as $(kx - \omega t)$

The frequency ω has units radians/s or similar & quantifies temporal frequency of oscillation.

$\nwarrow$ temporal frequency
 $\omega = ck$

k is analogous to ω !

A wave is an oscillation that is periodic in both

time and space. $k(x-ct)$ or $(kx-\omega t)$

Can also write solutions as, eg

$$\phi = \text{Re} \left\{ \phi_0 e^{i(kx-\omega t)} \right\}$$

$\uparrow$ constant

Recall $e^{i\theta} = \cos \theta + i \sin \theta$. Often we don't write $\text{Re}\{\}$.

For a 3D problem $\frac{\partial^2 \phi}{\partial t^2} = c^2 \nabla^2 \phi$

the solution is $\phi = \phi_0 e^{i(\vec{k} \cdot \vec{x} - \omega t)}$

$\uparrow$ position vector
 $\uparrow$ wave vector

Direction of $\vec{k}$ is the wave propagation direction.
Magnitude of $\vec{k}$ has same meaning as in 1D.

A wave is called longitudinal if it involves a vector quantity that oscillates in the same direction as the wave propagates.

A wave is called transverse if it involves a vector quantity that oscillates normal to the wave propagation direction. eg jump rope, $\vec{E} + \vec{B}$ electromagnetic

If $b_i = 0$, the Navier equation is solved by a wave of the form $\vec{u} = u_1 \hat{e}_1 = \hat{e}_1 \varepsilon \sin k(x_1 - ct)$.

$\uparrow$ displacement $\uparrow$ amplitude $\uparrow$ propagates in x_1 dir
oscillates in $\hat{e}_1$ dir

Longitudinal!

Now test the solution.

$$\vec{k} = k\hat{e}_1 + 0\hat{e}_2 + 0\hat{e}_3$$

$$\rho \frac{\partial^2 u_i}{\partial t^2} = (\lambda + \mu) \frac{\partial E_{ii}}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j} + \cancel{\rho b_i}$$

$$\frac{\partial u_1}{\partial t} = -\hat{e}_1 \epsilon k c_1 \cos k(x_1 - ct)$$

$$\frac{\partial^2 u_1}{\partial t^2} = -\hat{e}_1 \epsilon k^2 c_1^2 \sin k(x_1 - ct)$$

$$\frac{\partial u_1}{\partial x_1} = \hat{e}_1 \epsilon k \cos k(x_1 - ct)$$

$$\frac{\partial^2 u_1}{\partial x_1 \partial x_1} = -\hat{e}_1 \epsilon k^2 \sin k(x_1 - ct)$$

$$E_{jj} = \frac{\partial u_1}{\partial x_1} + \cancel{\frac{\partial u_2}{\partial x_2}} + \cancel{\frac{\partial u_3}{\partial x_3}}$$

$$\frac{\partial E_{jj}}{\partial x_j} = \frac{\partial^2 u_1}{\partial x_1}$$

$$\text{All other } \frac{\partial u_i}{\partial t} = 0$$

$$\text{All other } \frac{\partial u_i}{\partial x_j} = 0$$

Altogether, the x_1 component of the Navier eqn is

$$-\rho \hat{e}_1 \epsilon k^2 c_1^2 \sin k(x_1 - ct) = -(\lambda + \mu) \hat{e}_1 \epsilon k^2 \sin k(x_1 - ct)$$

$$-\mu \hat{e}_1 \epsilon k^2 \sin k(x_1 - ct)$$

$$c = \sqrt{\frac{\lambda + 2\mu}{\rho}}$$

speed of a longitudinal wave in an elastic solid

The solution does work, as long as this is true.

Conservation of energy for a generic continuous material:

$$\rho \frac{Du}{Dt} = T_{ij} D_{ij} - \frac{\partial q_i}{\partial x_i}$$

density ρ , internal energy per unit mass u , time t , stress T_{ij} , strain rate D_{ij} , heat flux q_i , space x_i

stress power: rate at which work is done on material element via deformation

Q: What if the only applied stress is an isotropic pressure? $T_{ij} = -p \delta_{ij}$

Then the stress power is $-p \delta_{ij} D_{ij} = -p D_{11} - p D_{22} - p D_{33}$

$$= -p D_{kk} = -p \frac{1}{V} \frac{\partial V}{\partial t} = -p \frac{\partial V}{\partial t} \frac{1}{V}$$

dilatation rate $\frac{\partial V}{\partial t}$, like $p \Delta V$ in intro thermo!

Review: Plane rotational waves

Navier: $\frac{\partial T_{ij}}{\partial x_j} = \lambda \delta_{ij} \frac{\partial E_{kk}}{\partial x_i} + 2\mu \frac{\partial E_{ij}}{\partial x_j}$

$$\rho \frac{\partial^2 u_i}{\partial t^2} = (\lambda + \mu) \frac{\partial E_{ii}}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j} + \rho f_i$$

One solution: $\vec{u} = \hat{e}_i \epsilon \sin k(x_i - ct)$

magnitude ϵ , wave number k , wave speed c iff $c = \sqrt{\frac{\lambda + 2\mu}{\rho}}$

"p waves" in geophysics.

Plane equivoluminal waves

Transverse! Another solution: $\vec{u} = \hat{e}_2 \epsilon \sin k(x_1 - ct)$

$$\frac{\partial u_2}{\partial t} = -kc\epsilon \cos k(x_1 - ct) \quad \frac{\partial u_1}{\partial t} = \frac{\partial u_3}{\partial t} = 0$$

Material elements displaced along $\hat{e}_2$ Wave propagates along $\hat{e}_1$

$$\frac{\partial u_2}{\partial x_1} = k\epsilon \cos k(x_1 - ct) \quad \frac{\partial u_2}{\partial x_2} = \frac{\partial u_2}{\partial x_3} = 0 \quad \frac{\partial^2 u_2}{\partial t^2} = -k^2 c^2 \epsilon \sin k(x_1 - ct)$$

$$\frac{\partial^2 u_2}{\partial x_1 \partial x_1} = -k^2 \epsilon \sin k(x_1 - ct)$$

$$E_{jj} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_1}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = 0 \quad \checkmark \quad \text{No volume change.}$$

Altogether, the $\hat{e}_2$ component of Navier eq'n becomes

$$-\rho k^2 c^2 \epsilon \sin k(x_1 - ct) = -\mu k^2 \epsilon \sin k(x_1 - ct)$$

$$c = \sqrt{\frac{\mu}{\rho}}$$

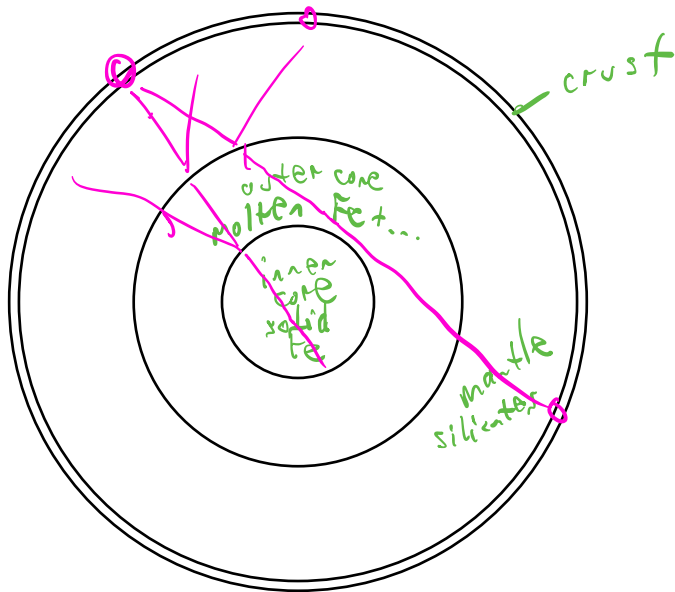
Solution works if & only if this is true. Wave speed!

Equivoluminal waves AKA shear waves ("s wave" in geophysics).

p wave: $c \approx \sqrt{\frac{\lambda + 2\mu}{\rho}}$ faster!

Waves in solids. Plane irrotational waves (longitudinal, pressure, $c = \sqrt{\frac{\lambda + 2\mu}{\rho}}$). Plane equivoluminal waves (transverse, $c = \sqrt{\frac{\mu}{\rho}}$, shear).

Seismology. No shear waves in fluids.



Wave polarization

Have considered longitudinal & transverse waves.

Polarization: rel'n b/w propagation direction & displacement direction.

linear (transverse) polarization: displacement $\perp$ to propagation and in 1 direction only

elliptical (transverse) polarization: superposition of displacements in both transverse directions

Elliptical polarization with equal amplitude in both directions and $\frac{\pi}{2}$ phase between them is "circular."

Wave reflection

Energy propagates in the wave propagation direction.

Energy can't enter vacuum.

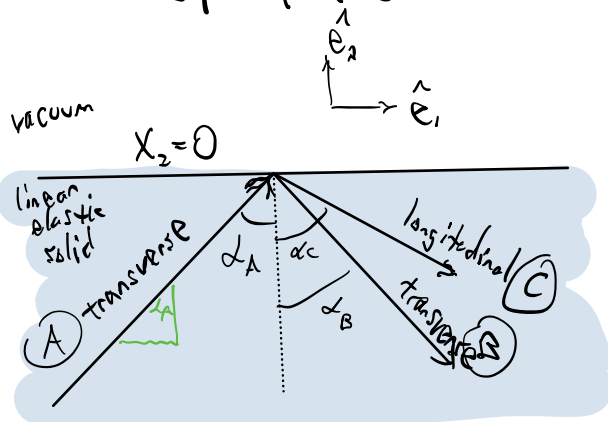
Must reflect!

traction-free

Boundary condition: stress-free

at $x_2=0$: $\vec{T} = \mathbb{T} \vec{n} \approx \mathbb{T} \hat{e}_2 = 0$

$$T_{12} = 0 \quad T_{22} = 0 \quad T_{32} = 0$$



Elastic solid: $T_{ij} = \lambda E_{kk} \delta_{ij} + 2\mu E_{ij}$

Assume (A) polarized such that there is no displacement in $\hat{e}_3$. Then $E_{32} = 0$ so $T_{32} = 0$. (happy cond ✓)

Now, $T_{12} = 0 \Rightarrow 0 = \lambda E_{kk} \delta_{12} + 2\mu E_{12}$ $E_{12} = 0$

Must write E_{12} in terms of wave motion.

$E_{12} = \frac{1}{2} \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right)$ where total displacement $\vec{u} = \vec{u}_A + \vec{u}_B + \vec{u}_C$
 (from waves)

(A) $\vec{u}_A = \hat{u}_A \hat{e}_A \sin(\vec{k}_A \cdot \vec{x} - \omega t)$

with $\vec{k}_A = k_A (\hat{e}_1 \sin \alpha_A + \hat{e}_2 \cos \alpha_A)$

The wave is transverse, so $\hat{u}_A \cdot \vec{k}_A = 0$

and $\hat{u}_A = \hat{e}_1 \cos \alpha_A - \hat{e}_2 \sin \alpha_A$. Altogether,

$\vec{u}_A = \hat{u}_A \sin(k_A x_1 \sin \alpha_A + k_A x_2 \cos \alpha_A - \omega t)$

δ_A

(B) Also transverse! By analogous reasoning,

$$\vec{u}_B = \varepsilon_B \hat{u}_B \sin(k_B x_1 \sin \alpha_B - k_B x_2 \cos \alpha_B - \omega t)$$

where $\hat{u}_B = \hat{e}_1 \cos \alpha_B + \hat{e}_2 \sin \alpha_B$, γ_B

(C) Longitudinal. Displacement // propagation.

$$\vec{u}_C = \varepsilon_C \hat{u}_C \sin(k_C x_1 \sin \alpha_C - k_C x_2 \cos \alpha_C - \omega t)$$

where $\hat{u}_C = \frac{\vec{k}_C}{k_C} = \hat{e}_1 \sin \alpha_C - \hat{e}_2 \cos \alpha_C$, γ_C

$$\text{Combining, } 0 = E_{12} = \frac{1}{2} \left(\varepsilon_A \cos^2 \alpha_A \cos \gamma_A - \varepsilon_B \cos^2 \alpha_B \cos \gamma_B - \varepsilon_C \cos \alpha_C \sin \alpha_C \cos \gamma_C - \varepsilon_A \sin^2 \alpha_A \cos \gamma_A + \varepsilon_B \sin^2 \alpha_B \cos \gamma_B - \varepsilon_C \cos \alpha_C \sin \alpha_C \cos \gamma_C \right)$$

$E_{12} = 0$ for any α_A . True only if $\gamma_A = \gamma_B = \gamma_C$.

If $\gamma_A = \gamma_C$ at $x_2 = 0$: $k_A \sin \alpha_A - \omega t = k_C \sin \alpha_C - \omega t$

$$\frac{\sin \alpha_A}{\sin \alpha_C} = \frac{k_C}{k_A}$$

define as n , the "index of refraction"

Since $c = \frac{\omega}{k}$, $n = \frac{\omega/c_C}{\omega/c_A} = \frac{c_A}{c_C} = \frac{\sqrt{\frac{\mu}{\rho}}}{\sqrt{\frac{\mu}{\rho + 2\mu}}} = \sqrt{\frac{\mu}{\rho + 2\mu}}$

Angle of reflected longitudinal wave is set by material properties (μ, ρ) & angle of incoming wave.

Now consider the requirement $\gamma_A = \gamma_B$ at $x_2 = 0$:

$$\frac{\sin \alpha_A}{\sin \alpha_B} = \frac{k_B}{k_A} = \frac{c_A}{c_B} = 1 \quad \text{both transverse!}$$

$$\alpha_A = \alpha_B$$

Angle of reflected transverse wave equals angle of incoming transverse wave.

Now consider the requirement $T_{22} = 0$ (and $E_{22} = 0$) at $x_2 = 0$. Following analogous calculation, get

$$\frac{E_B}{E_A} = \frac{\cos^2 2\alpha_A - n^2 \sin 2\alpha_A \sin 2\alpha_c}{\cos^2 2\alpha_A + n^2 \sin 2\alpha_A \sin 2\alpha_c}$$

$$\frac{E_c}{E_A} = \frac{n \sin 4\alpha_A}{\cos^2 2\alpha_A + n^2 \sin 2\alpha_A \sin 2\alpha_c}$$

Reflection coefficients

Fluids

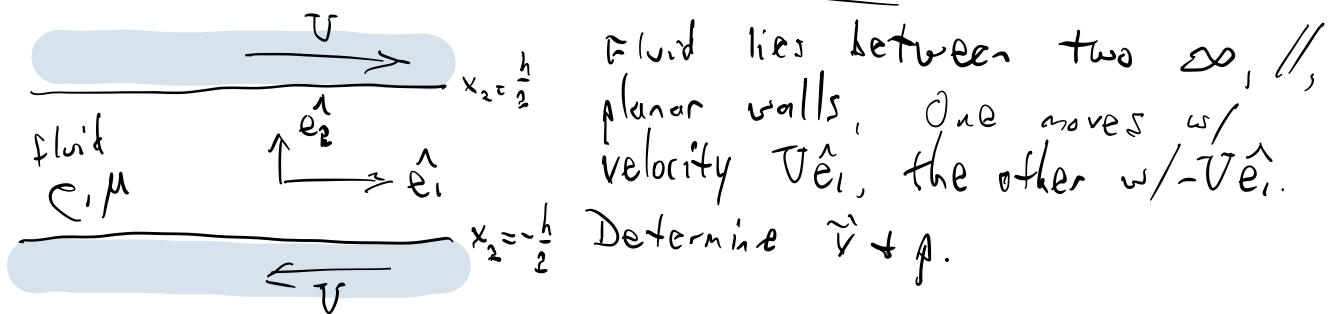
Newtonian fluid: $T_{ij} = -p \delta_{ij} + \lambda D_{kk} \delta_{ij} + 2\mu D_{ij}$

Momentum: $\rho \left(\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} \right) = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 v_i}{\partial x_j \partial x_j} + \rho b_i$
(if $D_{kk}=0$)

mass: $\frac{\partial v_i}{\partial x_i} = 0$

4 scalar equations
4 unknowns (v_i, p)
Could solve!

Example: shear flow (Couette flow)



Simplifying assumptions:

1. Steady flow: $\frac{\partial}{\partial t} = 0$. 2. 2D: $v_3 = 0$, $\frac{\partial}{\partial x_3} = 0$.

3. Fully developed: $\frac{\partial \vec{v}}{\partial x_1} = 0$

Mass: $\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} = 0$
(Note: v1 and v3 terms are crossed out with green lines, and '2D' is written in green next to the v3 term)
 $\int \{ \} dx_2$

Boundary conditions:
 $\vec{v} = U \hat{e}_1$ at $x_2 = \frac{h}{2}$
 $\vec{v} = -U \hat{e}_1$ at $x_2 = -\frac{h}{2}$
 $p = p_0$ at $\vec{x} = 0$.

From B.C.: $v_2 = \text{constant}$
 $v_2 = 0$ at wall $\Rightarrow v_2 = 0$

Momentum, $\hat{e}_1$: fully dev mass 2D

$$\rho \left(\cancel{\frac{\partial v_1}{\partial t}} + v_1 \cancel{\frac{\partial v_1}{\partial x_1}} + v_2 \cancel{\frac{\partial v_1}{\partial x_2}} + v_3 \cancel{\frac{\partial v_1}{\partial x_3}} \right) = - \frac{\partial p}{\partial x_2} + \mu \left(\cancel{\frac{\partial^2 v_1}{\partial x_1 \partial x_1}} + \cancel{\frac{\partial^2 v_1}{\partial x_2 \partial x_2}} + \cancel{\frac{\partial^2 v_1}{\partial x_3 \partial x_3}} \right)$$

$$\cancel{\mu} \frac{\partial p}{\partial x_2} = 0 \quad p \text{ doesn't vary w/ } x_2$$

$$\int \{ \quad \} dx_2$$

$$p = p_0 + \cancel{f(x_1)}$$

Momentum, $\hat{e}_1$: fully dev mass 2D

$$\rho \left(\cancel{\frac{\partial v_1}{\partial t}} + v_1 \cancel{\frac{\partial v_1}{\partial x_1}} + v_2 \cancel{\frac{\partial v_1}{\partial x_2}} + v_3 \cancel{\frac{\partial v_1}{\partial x_3}} \right) = \cancel{\frac{\partial p}{\partial x_1}} + \mu \left(\cancel{\frac{\partial^2 v_1}{\partial x_1 \partial x_1}} + \cancel{\frac{\partial^2 v_1}{\partial x_2 \partial x_2}} + \cancel{\frac{\partial^2 v_1}{\partial x_3 \partial x_3}} \right)$$

$$\cancel{\frac{\partial^2 v_1}{\partial x_2 \partial x_2}} = \cancel{\frac{1}{\mu} \frac{\partial p}{\partial x_1}}$$

No pressure gradient imposed.

$$\int \{ \quad \} dx_2$$

$$\int \left\{ \frac{\partial v_1}{\partial x_2} = C_1 \right\} dx_2$$

$$v_1 = C_1 x_2 + C_2$$

BC:

$$U = C_1 \frac{h}{2} + C_2 \quad (1)$$

$$-U = -C_1 \frac{h}{2} + C_2 \quad (2)$$

$$(1) + (2): 0 = 2C_2$$

$$(1) - (2): 2U = C_1 h$$

$$C_1 = 2U/h$$

Altogether, $v_1 = 2U \frac{x_2}{h}$



Example: pressure-driven flow b/w plates (Poiseuille)

$x_2 = h$

Applying a constant, known pressure gradient $\partial p / \partial x_1 < 0$.

ρ, μ $\hat{e}_2$ $\hat{e}_1$ $x_2 = 0$

Walls don't move.

B.C.s: $\vec{v} = 0$ at $x_2 = 0 + h$.

Same simplifying assumptions: $\frac{\partial}{\partial t} = \frac{\partial}{\partial x_3} = \frac{\partial \vec{v}}{\partial x_1} = v_3 = 0$

Mass: $\vec{v}_2 = 0$ as before.

Momentum, $\hat{e}_2$: $\rho \left(\dots \cancel{v_2 \frac{\partial v_2}{\partial x_2}} \right) = - \frac{\partial p}{\partial x_2} + \mu \left(\cancel{v_2} \right)$

p doesn't vary w/ x_2 .

constant known! As before.

Momentum, $\hat{e}_1$: $\rho \left(\cancel{v_1 \frac{\partial v_1}{\partial x_2}} \right) = - \frac{\partial p}{\partial x_1} + \mu \left(\frac{\partial^2 v_1}{\partial x_2^2} \right)$

$$\int \left\{ \frac{\partial^2 v_1}{\partial x_2^2} = \frac{1}{\mu} \frac{\partial p}{\partial x_1} \right\} dx_2$$

$$\int \left\{ \frac{\partial v_1}{\partial x_2} = \frac{1}{\mu} \frac{\partial p}{\partial x_1} x_2 + C_1 \right\} dx_2$$

$$v_1 = \frac{1}{2\mu} \frac{\partial p}{\partial x_1} x_2^2 + C_1 x_2 + C_2$$

B.C.s:

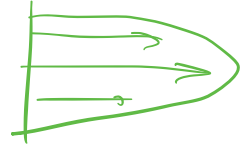
$$0 = \frac{1}{2\mu} \frac{\partial p}{\partial x_1} h^2 + C_1 h + C_2 \quad (3)$$

$$0 = 0 + 0 + C_2$$

from (3): $C_1 = -\frac{1}{2\mu} \frac{\partial p}{\partial x_1} h$

Altogether, $V_1 = \frac{1}{2\mu} \frac{\partial p}{\partial x_1} x_2^2 - \frac{1}{2\mu} \frac{\partial p}{\partial x_1} h x_2$

$$V_1 = \frac{1}{2\mu} \frac{\partial p}{\partial x_1} x_2 (x_2 - h)$$



Often write $\frac{\mu}{\rho}$. "dynamic viscosity" Define $\nu \equiv \frac{\mu}{\rho}$. "kinematic viscosity"

$$T_{12} = 2\mu D_{12}$$

$\text{Pa} \quad \text{Pa}\cdot\text{s} \quad \frac{1}{\text{s}}$

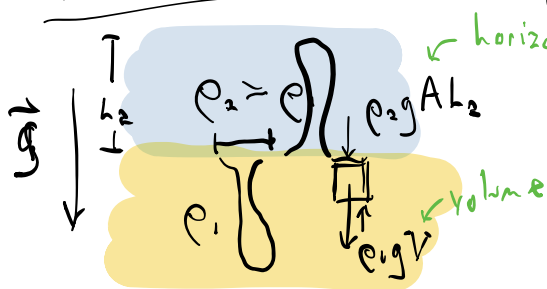
$$\frac{\text{Pa}\cdot\text{s}}{\text{kg/m}^3} = \frac{\text{kg}\frac{\text{m}}{\text{s}}\frac{1}{\text{m}^3}}{\text{kg}\frac{1}{\text{m}^3}} = \frac{\text{m}^2}{\text{s}}$$

Kinematic diffusivity ν is the momentum diffusivity the rate of increase of area "covered by" an injection of momentum.

$$\text{Navier-Stokes: } \rho \left(\frac{\partial \vec{v}}{\partial t} + \underbrace{(\vec{v} \cdot \nabla) \vec{v}}_{\text{nonlinear!}} \right) = - \nabla p + \mu \nabla^2 \vec{v}$$

Last time, assumptions (steady, 2D, fully-developed) made nonlinear term go to 0! Key for solving.

A bit about the Rayleigh-Taylor instability



One equilibrium solution:

No horizontal forces, no motion, denser fluid stays above the less-dense fluid.

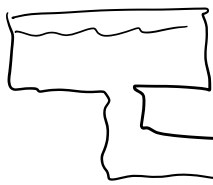
But any perturbation causing even teeny horizontal forces grows exponentially over time. Unstable equilib.! Denser fluid ends up on bottom, after complicated flow + mixing.

Dimensional Analysis & Modeling

Big idea: Write equations of motion in dimensionless form in order to build experimental/numerical models whose results can be generalized past a specific, single case. (Also keep numbers $\sim O\{1\}$ for simulations.)

Buckingham's Π theorem (1914): Any physical problem in which there are n relevant quantities expressed in terms of r fundamental units can be written in terms of $n-r$ dimensionless quantities.

Ex: Pipe flow. Osborne Reynolds, 1883: What makes pipe flow turbulent?



Experiments showed that 3 quantities mattered: diameter L , mean speed U and viscosity ν . $\frac{m}{s}$

Can write this problem w/ $3-2=1$ dimensionless qty. $r=2$ base units

$$\frac{LU}{\nu} = Re \quad \text{"Reynolds number"}$$

Osborne found turbulence for $Re \gtrsim 2060$.

Can devise dimensionless q'tys ad-hoc, but they're encoded in the equations.

Momentum: $\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \vec{v}$ (1) Mass: $\nabla \cdot \vec{v} = 0$ (2)

We know the important q'tys: L, U, ν . Plan: write

(1) & (2) in dimensionless form, via change of variables using L, U, ν .

Def: $\tilde{\vec{v}} \approx \frac{\vec{v}}{U}$, $\tilde{p} = \frac{p}{\rho U^2}$, $\tilde{\nabla} = L \nabla$, $\frac{\partial}{\partial \tilde{t}} = \frac{L}{U} \frac{\partial}{\partial t}$. Into (1):

$$\left\{ \frac{U^2}{L} \frac{\partial \tilde{\vec{v}}}{\partial \tilde{t}} + \frac{U^2}{L} (\tilde{\vec{v}} \cdot \tilde{\nabla}) \tilde{\vec{v}} = -\frac{U^2}{L} \tilde{\nabla} \tilde{p} + \frac{\nu U}{L^2} \tilde{\nabla}^2 \tilde{\vec{v}} \right\} \frac{L}{U^2}$$

$$\frac{\partial \tilde{\mathbf{v}}}{\partial t} + (\tilde{\mathbf{v}} \cdot \nabla) \tilde{\mathbf{v}} = - \nabla \tilde{p} + \underbrace{\frac{\nu}{LU}}_{\text{Re}} \nabla^2 \tilde{\mathbf{v}}$$

Can also write mass eq'n in dimless form. Get $\tilde{\mathbf{v}} \cdot \nabla = 0$
 No additional dimensionless quantities.

Can design & run models w/ different size, fluid, etc than the system of our interest & make good predictions as long dimless params match.

Waves in fluids

Many kinds, including sound, surface gravity waves, capillary waves (surface tension), internal gravity waves (smooth density variation), inertial waves (Coriolis force, AKA Rossby waves). No shear waves!

Consider acoustic (sound) waves. Longitudinal.
 Require compressibility. Make a small correction to incompressible mass & momentum eq'ns:

1. Let v_i be small.

2. Let $p = p_0 + p'$ $p = p_0 + p'$
mean fluctuation assuming fluctuations are small

Use in momentum eq'n:

$$\frac{\partial v_i}{\partial t} + \underbrace{v_j \frac{\partial v_i}{\partial x_j}}_{\text{second-order small}} = - \frac{1}{\underbrace{p_0 + p'}_{\text{constant}}} \frac{\partial p_0}{\partial x_i} - \frac{1}{\underbrace{p_0 + p'}_{\text{constant}}} \frac{\partial p'}{\partial x_i} + \underbrace{\nu \frac{\partial^2 v_i}{\partial x_j \partial x_j}}_{\text{special case: inviscid } (\nu=0)}$$

Last time: Thinking about slow flow (v_i small) in a Newtonian fluid with weak compressibility such that $p = p_0 + p'$ and $\rho = \rho_0 + \rho'$, in the case where viscous effects are negligible. Momentum eqn became

$$\rightarrow \frac{\partial v_i}{\partial t} = -\frac{1}{\rho_0} \frac{\partial p'}{\partial x_i} \quad (\text{A})$$

Conservation of mass gives $\rho \frac{\partial v_i}{\partial x_i} + v_i \frac{\partial \rho}{\partial x_i} + \frac{\partial \rho}{\partial t} = 0$

$$\rho_0 \frac{\partial v_i}{\partial x_i} + \cancel{\rho' \frac{\partial v_i}{\partial x_i}} + v_i \frac{\partial \rho_0}{\partial x_i} + v_i \frac{\partial \rho'}{\partial x_i} + \cancel{\frac{\partial \rho_0}{\partial t}} + \frac{\partial \rho'}{\partial t} = 0$$

$$\rightarrow \frac{\partial v_i}{\partial x_i} = -\frac{1}{\rho_0} \frac{\partial \rho'}{\partial t} \quad (\text{B})$$

$$\text{Combine } \frac{\partial}{\partial x_i} (\text{A}) \text{ and } \frac{\partial}{\partial t} (\text{B}) \text{ to get } -\frac{1}{\rho_0} \frac{\partial^2 p'}{\partial x_i \partial x_i} = -\frac{1}{\rho_0} \frac{\partial^2 \rho'}{\partial t^2} \quad (\text{C})$$

Density varies with pressure! For convenient math, write $p = p(\rho)$. Taylor expansion around $\rho = \rho_0$:

$$p = p_0 + (\rho - \rho_0) \left. \frac{\partial p}{\partial \rho} \right|_{\rho_0} + \frac{1}{2} (\rho - \rho_0)^2 \left. \frac{\partial^2 p}{\partial \rho^2} \right|_{\rho_0} + \dots$$

$$\text{To first order, } \underbrace{p - p_0}_{p'} = \underbrace{(\rho - \rho_0)}_{\rho'} \left. \frac{\partial p}{\partial \rho} \right|_{\rho_0}$$

$$\rho' = \frac{p'}{\left. \frac{\partial p}{\partial \rho} \right|_{\rho_0}}$$

$$\text{into (C): } \frac{\partial^2 p'}{\partial t^2} = \left. \frac{\partial p}{\partial \rho} \right|_{\rho_0} \frac{\partial^2 p'}{\partial x_i \partial x_i}$$

A wave equation!

units $\frac{m^2}{s^2}$ - square of wave speed

This equation admits wave solutions like

$$p = \hat{p} e^{i(k \cdot \vec{x} - \omega t)} \quad \text{or} \quad p = \hat{p} \sin(k_i x_i - \omega t)$$

$$\text{or } p = \hat{p} \sin k_i (x_i - ct)$$

Longitudinal! Sound waves.

For a linear material, bulk modulus = $\frac{\text{pressure}}{\text{dilatation}}$

$$k = \frac{p'}{\rho_0 \frac{dV}{V}}$$

$$\text{So wave speed } c = \sqrt{\left. \frac{\partial p}{\partial \rho} \right|_{\rho_0}} \approx \sqrt{\frac{k}{\rho_0}}$$

Compare to longitudinal wave speed in elastic solid $c = \sqrt{\frac{k + \frac{4}{3}\mu}{\rho}}$

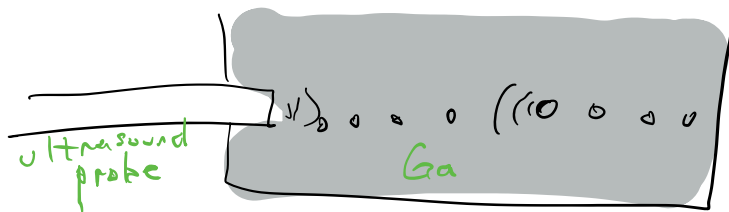
Longitudinal wave speed in a fluid matches that for a solid with zero shear modulus.

Water: $k = 2.15 \text{ GPa}$, $\rho = 1000 \frac{\text{kg}}{\text{m}^3} \Rightarrow c = \sqrt{\frac{k}{\rho}} = 1460 \frac{\text{m}}{\text{s}}$

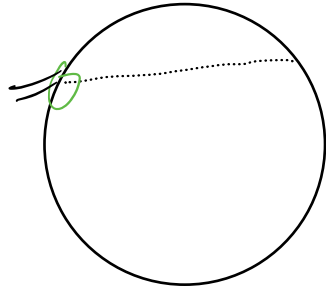
Measured sound speed $1498 \frac{\text{m}}{\text{s}}$.

Air $\sim 350 \frac{\text{m}}{\text{s}}$ @ STP

A strictly incompressible fluid allows no volume change, so $k = \infty$ and $c = \infty$. Mechanical perturbations are felt everywhere, immediately. Unrealistic.



Get echoes at discontinuities of acoustic impedance ρc



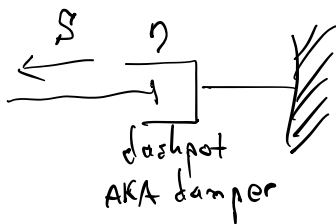
Viscoelasticity ch.7 in Lai

Elastic solids have stress linearly related to strain: $T_{ij} = \lambda E_{kk} \delta_{ij} + 2\mu E_{ij}$
 Newtonian fluids have stress linearly related to strain rate: $T_{ij} = -p \delta_{ij} + \lambda D_{kk} \delta_{ij} + 2\mu D_{ij}$
 Materials for which stress depends both on strain and on strain rate are "viscoelastic".

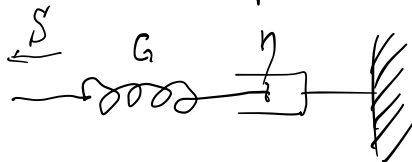
Consider analogous, discrete mechanical elements:



strain ϵ , force $S = G \epsilon$. stress $\propto$ strain.
 Analogous to elastic solid.



strain ϵ , force $S = \eta \frac{\partial \epsilon}{\partial t}$. stress $\propto$ strain rate
 Analogous to Newtonian fluid. ↑ damping coeff



Same force on both spring & damper
 b/c connected in series.

$$\textcircled{1} S = G \epsilon_e \quad \textcircled{2} S = \eta \frac{\partial \epsilon_d}{\partial t} \leftarrow \text{dumper strain}$$

↑ spring strain

In series, so strains add: $\frac{\partial}{\partial t} \epsilon = \epsilon_e + \epsilon_d \}$ total strain discrete
+ discrete

use $\frac{\partial}{\partial t}$ (1) and (2) : $\frac{\partial \varepsilon}{\partial t} = \frac{1}{G} \frac{\partial S}{\partial t} + \frac{1}{\eta} S$

Define $\lambda = \frac{\eta}{G}$ + rearrange to get $S + \lambda \frac{\partial S}{\partial t} = \eta \frac{\partial \varepsilon}{\partial t}$

The continuous material analogous to this ^{discrete} system is viscoelastic and has constitutive rel'n

$$\tau_{ij} = -p \delta_{ij} + S_{ij} \quad \text{where} \quad S_{ij} + \lambda \frac{\partial S_{ij}}{\partial t} = 2\mu D_{ij}$$

A "Maxwell fluid"

Viscoelasticity, Maxwell fluid $T_{ij} = -p\delta_{ij} + S_{ij}$
 with $S_{ij} + \lambda \frac{\partial S_{ij}}{\partial t} = 2\mu D_{ij}$
 ↑ p_a ↑ shear viscosity μ Pa-s

Thought experiment: oscillating strain

$x_2 = h$ $u_1(h) = u_0 e^{i\omega t}$
 ← →
 Maxwell fluid x_1

Last time, solved for stress:

$$S_{12} = e^{i\omega t} \frac{i\omega\mu u_0}{h(1+i\omega\lambda)} \quad (\text{real part})$$

$x_2 = 0$

What's the shear modulus for a Maxwell fluid in oscillation?

$$\begin{aligned} \text{shear modulus} &= \frac{\text{shear stress}}{\text{shear strain}} = \frac{S_{12}}{2E_{12}} = \frac{S_{12}}{2 \cdot \frac{1}{2} \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right)} \\ &= \frac{S_0}{u_0/h} = \frac{i\omega\mu}{1+i\omega\lambda} \left(\frac{1-i\omega\lambda}{1-i\omega\lambda} \right) \\ &= \underbrace{\frac{\omega^2\mu\lambda}{1+\omega^2\lambda^2}}_{G'} + i \underbrace{\frac{\omega\mu}{1+\omega^2\lambda^2}}_{G''} = G^* \end{aligned}$$

If $G^* = \text{Re}\{G^*\} = G'$, no phase shift b/w stress & strain.

If $G^* = \text{Im}\{G^*\} = G''$, 90° phase shift " " " "

If G^* complex, phase shift could be anything.

Suppose $\lambda = 0$ (instantaneous relaxation; viscosity dominates, fluid-like). Then $G' = 0$ and $G^* = iG'' = \omega\mu$.

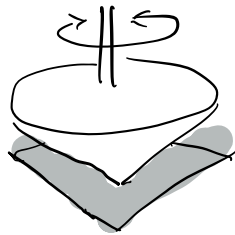
G'' tells us about viscous effects. Called "loss modulus" b/c viscosity damps motion.

Suppose $\lambda \rightarrow \infty$ (very slow relaxation; elasticity dominates; solid-like). Then $G'' = 0$ (second-order small) and

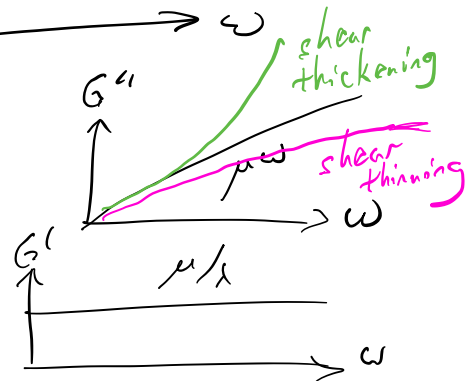
$$G' = \frac{\omega^2 \mu \lambda}{1 + \omega^2 \lambda^2} \frac{1}{\lambda} = \frac{\omega^2 \mu}{1 + \omega^2 \lambda^2} = \frac{\mu}{\lambda}$$

G' tells us about elasticity. Called "storage modulus" in analogy to storage of potential energy in a spring.

A rheometer does this experiment.



G' G''



Newtonian fluid: $G' = 0$ and

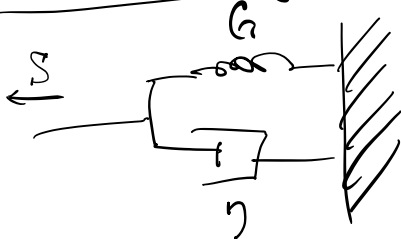
Elastic solid: $G'' = 0$ and

$\sigma_{12} = \frac{u_0}{h} (G' + i G'')$ as above can also be

written $\sigma_{12} = \frac{u_0}{h} |G^*| e^{i(\omega t + \delta)}$

where $\delta = \tan^{-1} \frac{G''}{G'}$.

Kelvin-Voigt solids



maxwell: spring & damper in series
K-V : " " parallel

Strain on spring equals strain on damper (parallel).

$$S = S_e + S_v = G\varepsilon + \eta \frac{\partial \varepsilon}{\partial t}$$

$\uparrow$
spring
force
 $\uparrow$
damper
force
 $\uparrow$
strain

Kelvin-Voigt solids have analogous constitutive relation:

$$T_{ij} = -p\delta_{ij} + S_{ij} \quad S_{ij} = G' E_{ij} + 2\mu \dot{D}_{ij}$$

$\uparrow$
storage
modulus
 $\uparrow$
shear viscosity

Thought experiment: Sudden stress

$S'_{ij} = 0$ for $t < 0$. For $t > 0$, $S_{12} = S_{21} = S_0$, all other $S'_{ij} = 0$.

Determine the strain.

$$S_{ij} = G' E_{ij} + 2\mu \dot{D}_{ij}$$

~~$$S_{12} = G' E_{12} + 2\mu \dot{D}_{12} = G' \frac{1}{2} \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) + 2\mu \frac{1}{2} \left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} \right)$$~~

$$S'_{12} = G' E_{12} + 2\mu \frac{\partial E_{12}}{\partial t}$$

Solve homogeneous eq'n first: $\frac{\partial E_{12}}{\partial t} = -\frac{G'}{2\mu} E_{12}$

Sol'n is $E_{12} = C e^{-\lambda t}$

Substitute: ~~$\lambda C e^{-\lambda t} = -\frac{G'}{2\mu} C e^{-\lambda t}$~~

$$\lambda = \frac{G'}{2\mu}$$

Now find particular solution. As $t \rightarrow \infty$, expect approach to a steady state ($\frac{\partial E_{12}}{\partial t} = 0$). $S_0 = G' E_{12}$

$$E_{12} = \frac{S_0}{G'}$$

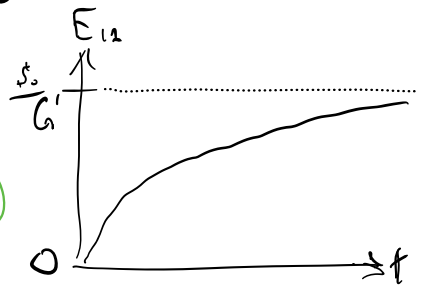
Together, $E_{12} = C e^{-\frac{G'}{2\mu} t} + \frac{S_0}{G'}$

Initial condition: $E_{12} = 0$ at $t=0$ and z

$$0 \approx C \cancel{e^0} + \frac{S_0}{G'}$$

$$G = - \frac{S_0}{G'}$$

Altogether, $\Xi_{12} = \frac{S_0}{G'} \left(1 - e^{-\frac{G'}{2V} t} \right)$



At long times, get constant strain (no motion) like a solid. Contrast w/ Maxwell fluid.

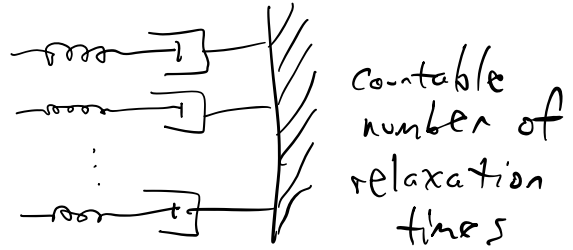
Characteristic time scale is $\frac{G'}{2\mu}$.

Analogous to RC for a circuit.

K-V is a low-pass filter for strain.

Maxwell, high-pass

Generalized Maxwell fluids with continuous relaxation spectra. Discrete case:



Continuous case: in general, any time scale can play a role in the relaxation.

$$\Sigma = \int_{-\infty}^t 2\phi(t-t') \mathbb{D}(t') dt'$$

$$\text{with } \phi(t) = \int_0^{\infty} \underbrace{\frac{H(\lambda)}{\lambda}}_{\text{relaxation spectrum}} e^{-t/\lambda} d\lambda$$

Now, relate $H(\lambda)$ to measurable quantities G' & G'' .

$$\text{Recall } G^* = \underbrace{G'}_{\text{shear mod}} + i \underbrace{G''}_{\text{storage mod}} \underbrace{}_{\text{loss mod}}$$

For a simple Maxwell fluid (discrete, $N=1$ relaxation time),

$$G^* = \frac{i\omega\mu}{1+i\omega\lambda} \quad \text{Write in terms of } \phi(t), \text{ in order to accommodate continuous spectra.}$$

$$\text{Can be written } G^* = i\omega \int_{-\infty}^t \phi(t-t') e^{-i\omega(t-t')} dt'$$

$$\text{Ex: Simple Maxwell fluid w/ } \phi = \frac{\mu}{\lambda} e^{-\frac{t-t'}{\lambda}}$$

$$\begin{aligned} \therefore G^* &= i\omega \int_{-\infty}^t \frac{\mu}{\lambda} e^{-\frac{t-t'}{\lambda} - i\omega(t-t')} dt' \\ &= i\omega \frac{1}{\frac{1}{\lambda} + i\omega} e^{-\frac{t-t'}{\lambda} - i\omega(t-t')} \bigg|_{t'=-\infty}^{t'=t} \end{aligned}$$

$$= \frac{i\omega\mu}{1+i\omega\lambda} \left(e^{0-i\omega\cdot 0} - e^{-\frac{t+\infty}{\lambda} - i\omega(t+\infty)} \right)$$

Continuous spectrum: $\phi(t) = \int_0^\infty \frac{H(\lambda)}{\lambda} e^{\frac{t}{\lambda}} d\lambda$

$$G^* = i\omega \int_{-\infty}^t \int_0^\infty \frac{H(\lambda)}{\lambda} e^{-\frac{t-t'}{\lambda}} d\lambda e^{-i\omega(t-t')} dt'$$

$$= i\omega \int_0^\infty \frac{H(\lambda)}{\lambda} \int_{-\infty}^t e^{-\frac{t-t'}{\lambda} - i\omega(t-t')} dt' d\lambda$$

$= \frac{\lambda}{1+i\omega\lambda}$ from above

$$G^* = i\omega \int_0^\infty \frac{H(\lambda)}{1+i\omega\lambda} d\lambda$$

Get S_{ij} , use in $T_{ij} = -p\delta_{ij} + S_{ij}$

use T_{ij} in $\frac{\partial T_{ij}}{\partial x_j} = \rho a_i + \rho b_j$ momentum eq'n

Poroelectricity & flow through biological tissues

Glymphatic system: Circulation of water-like fluids (cerebrospinal fluid & interstitial fluid) thru spaces in the brain. Fluid is well-modeled as Newtonian & incompressible. Speeds $\sim 10^{-5}$ m/s or less. Vessel width $\sim 10^{-5}$ m. Vessel/spaces have complicated or less.

shapes move, & are soft.

$$Re = \frac{UL}{\nu} = \frac{(10^{-5} \frac{m}{s})(10^{-5} m)}{0.7 \times 10^{-6} \frac{m^2}{s}} \sim 10^{-4}$$

$\downarrow$ speed $\downarrow$ size
 $\uparrow$ kinematic viscosity

Water

$$\rho = 1000 \frac{kg}{m^3}$$

$$\nu = 10^{-6} \frac{m^2}{s} \text{ c.R.T.}$$

$$\mu = 10^{-3} Pa \cdot s$$

Totally laminar!

Expect turbulence only for $Re \gtrsim 10^3$. But geometry is complicated!

Need tools to deal with flow through big networks of complicated spaces.

Hydraulic Resistance

Analogous to electrical resistance.

I current (charge/s)

Q volume flow rate (fluid volume)

ΔV voltage (potential energy / charge)

Δp pressure (potential energy / volume)

Define electrical resistance $R = \frac{\Delta V}{I}$

Conveniently, for many devices, R is a constant.

We want to define hydraulic resistance $R = \frac{\Delta p}{Q}$

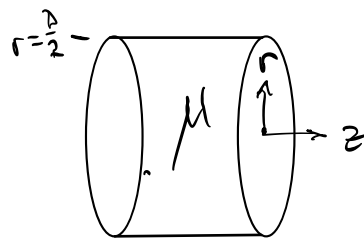
Hydraulic resistance is also constant as long as the flow experiences no instabilities.

True if Re small. $\checkmark$

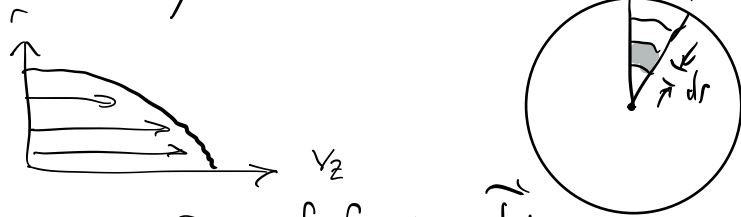
One example: pipe flow.

Flow driven by pressure gradient $\frac{\partial p}{\partial z}$.

Boundary condition: $\vec{v} = 0$ at $r = \frac{3}{2}$



Symmetry assumptions: steady ($\frac{\partial}{\partial t} = 0$), axially symmetric
 AKA axisymmetric ($\frac{\partial}{\partial \phi} = 0$, $v_\phi = 0$), fully developed
 ($\frac{\partial \vec{v}}{\partial z} = 0$). Solve $\vec{\nabla} \cdot \vec{v} = 0$ and $\rho \frac{D\vec{v}}{Dt} = -\vec{\nabla} p + \mu \nabla^2 \vec{v}$
 together to get $\vec{v} = -\frac{1}{4\mu} \frac{\partial p}{\partial z} \left(\left(\frac{D}{2}\right)^2 - r^2 \right) \hat{e}_z$.



Volume flow rate is $Q = \int \int \vec{v} \cdot d\vec{A}$
 $A \leftarrow$ cross-sectional area

Here, $Q = \int_0^{2\pi} \int_0^{\frac{D}{2}} -\frac{1}{4\mu} \frac{\partial p}{\partial z} \left(\left(\frac{D}{2}\right)^2 - r^2 \right) \hat{e}_z \cdot \vec{r} dr d\phi \hat{e}_z$

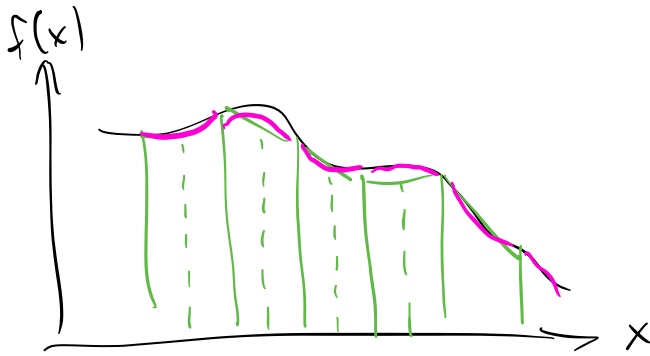
$$Q = 2\pi \left(-\frac{1}{4\mu} \frac{\partial p}{\partial z} \right) \int_0^{\frac{D}{2}} \left(\left(\frac{D}{2}\right)^2 r - r^3 \right) dr$$

$$\left(\frac{1}{2} \left(\frac{D}{2}\right)^2 r^2 - \frac{1}{4} r^4 \right) \Big|_0^{\frac{D}{2}} = \frac{1}{4} \left(\frac{D}{2}\right)^4$$

$$Q = -\frac{\pi}{8\mu} \frac{\partial p}{\partial z} \left(\frac{D}{2}\right)^4 = -\frac{\pi}{128\mu} \frac{\partial p}{\partial z} D^4 \leftarrow \text{scales as } D^4!$$

So the hydraulic resistance is

$$R = \frac{\Delta p}{Q} = \frac{\frac{\partial p}{\partial z} L}{-\frac{\pi}{128\mu} \frac{\partial p}{\partial z} D^4} = -\frac{128\mu L}{\pi D^4}$$



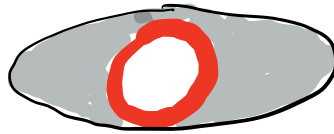
Hydraulic resistance: $R = \frac{\Delta p}{Q}$

For laminar ^{cylindrical} pipe flow $R = \frac{128 \mu L}{\pi \Delta^4}$

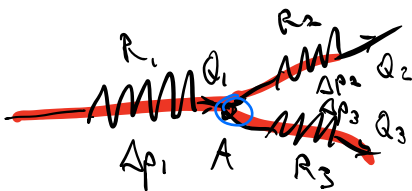
For cross-sectional shapes other than cylinders can again calculate R (analytically or numerically). Always find

$$R \propto \frac{\mu L}{A^2}$$

$A^2 \leftarrow$ cross-sectional area



Hydraulic resistance enables simple models of big flow networks. Consider a bifurcation:



Know that

$$\begin{aligned} Q_1 &= R_1 \Delta p_1 \\ Q_2 &= R_2 \Delta p_2 \\ Q_3 &= R_3 \Delta p_3 \end{aligned}$$

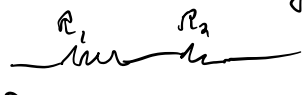
Cons. of mass: All flow rates into (A) must sum to zero.

$$Q_1 = Q_2 + Q_3$$

Analogous to conservation of charge (Kirchoff's current law).

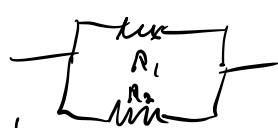
Cons. of energy: Total pressure drop around any closed loop is zero.

Analogous to conservation of energy (Kirchhoff's voltage law)

Consider a series connection: 

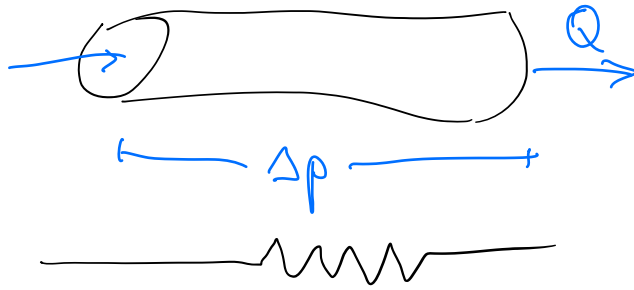
Same flow rate thru (1) & (2)

Total pressure drop is sum of $\Delta p_1 + \Delta p_2$

Consider a parallel connection: 

Same pressure drop. Flow rates sum.

Have been considering hydraulic resistance.

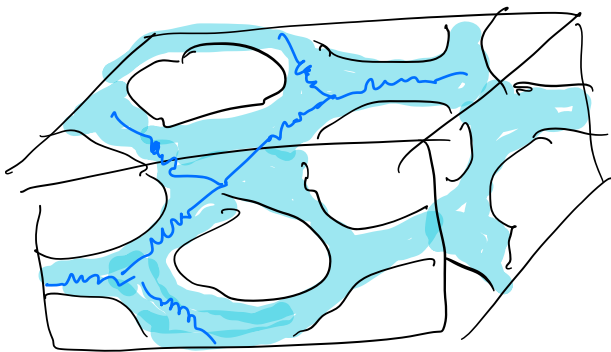


$$R = \frac{\Delta p}{Q}, \text{ const if laminar}$$

Can combine to make complex flow networks.

Porous media

Materials harboring even more complicated pathways.



Grains of sand w/ H_2O between
Rocks with oil between

Brewing coffee

Interstitial spaces in tissue

Like a generalization of a complex hydraulic resistance network. Idea: too much trouble to capture full details of microscale structure (shape of every grain), so capture average effect on flow.

Henry Darcy, mid-1800s, empirically found

Darcy's Law

$$\bar{q} = - \frac{k}{\mu} \nabla p$$

$\bar{q}$ (m/s) ← mean flow thru porous medium
 k ← permeability (m^2)
 μ (Pa·s) ← viscosity
 ∇p ← pressure

Always, to first order,
 $k \propto$ smallest cross-sectional area (pore size) thru which fluid is forced.

A bit like resistance an inverse resistance...

$$R = \frac{\Delta p}{Q} = \frac{\Delta p}{\int \vec{q} \cdot d\vec{A}} = \frac{\Delta p}{\frac{k}{\mu} \nabla p \cdot \vec{A}} = \frac{\Delta p}{\frac{k}{\mu} \frac{\Delta p}{L} A} = \frac{\mu L}{kA}$$

$$\text{So } k = \frac{\mu L}{RA}$$

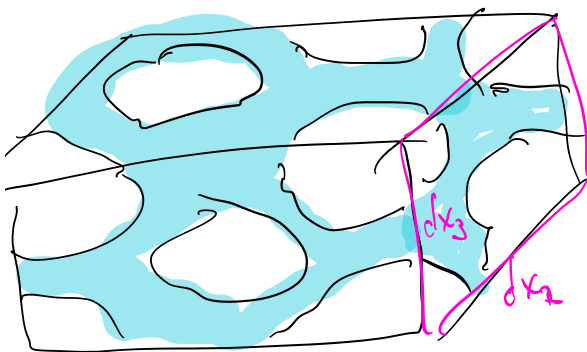
Can calculate k by hand for simple geometry, eg pipe flow:

$$\vec{V} = -\frac{1}{4} \mu \frac{\partial p}{\partial z} \left(\left(\frac{D}{2} \right)^2 - r^2 \right) \hat{e}_z$$

$$\vec{q} = \frac{1}{A} \int \vec{V} \cdot d\vec{A} = \frac{1}{\pi D^2/4} \int_0^{2\pi} \int_0^{\frac{D}{2}} \vec{V} \cdot \hat{e}_z r \, dr \, d\phi \hat{e}_z = -\frac{1}{8\mu} \frac{\Delta p}{L} \left(\frac{D}{2} \right)^2$$

$$\text{Compare to } \vec{q} = -\frac{k}{\mu} \nabla p \text{ \& see } k = \frac{D^2}{32}$$

More about the meaning of $\vec{q}$



Suppose we apply a pressure gradient and observe fluid exiting this mat'l element in the $\hat{e}_1$ dir w/ flow rate Q . If the

area is $dx_2 dx_3$ then $q_1 = \frac{Q}{dx_2 dx_3}$. Likewise

the components $q_2 + q_3$ are volume/second/cross-sectional area in those directions.

$\vec{q}$ is average flow speed across a slice of porous medium. Known as "superficial velocity" or "specific discharge" or "apparent velocity".

Could also calculate the average velocity considering only the pores (not grains, which block flow), $\langle \vec{V} \rangle$ known as "microscale velocity", "seepage velocity", "filtration velocity".

$$\vec{q} = \phi \langle \vec{V} \rangle$$

$\uparrow$ porosity
 pore volume / total volume fluid volume / total volume

Darcy's law fails unless flow is laminar.

$$\vec{q} = -\frac{k}{\mu} \vec{\nabla} p.$$

Mathematically appropriate boundary condition is free slip (no penetration): $\vec{q} \cdot \hat{n} = \vec{V}_{\text{wall}} \cdot \hat{n}$

$\uparrow$
wall normal

If the fluid is incompressible, $\vec{\nabla} \cdot \vec{q} = 0$.

Use Darcy: $\vec{\nabla} \cdot \left(-\frac{k}{\mu} \vec{\nabla} p \right) = 0$

$$-\frac{k}{\mu} \nabla^2 p = 0 \quad \text{Simple, linear PDE!}$$

Let's consider 2D flows. What are the solutions of $\nabla^2 p = 0$?

1. $p = \text{constant} \Rightarrow \vec{q} = -\frac{k}{\mu} \vec{\nabla} \text{constant} = 0$

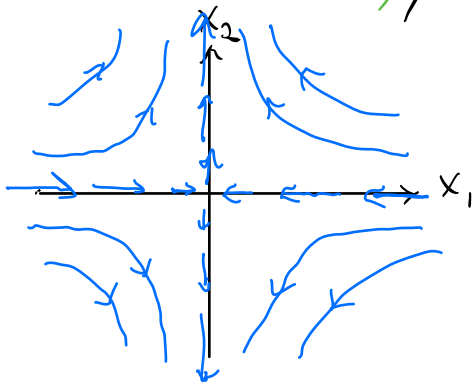
2. linear: $p = \alpha_1 x_1 + \alpha_2 x_2 \Rightarrow \vec{q} = -\frac{k}{\mu} (\alpha_1 \hat{e}_1 + \alpha_2 \hat{e}_2)$
uniform flow.

3. $p = \frac{\rho}{2} (x_1^2 - x_2^2)$

Note $\frac{\partial p}{\partial x_1} = \frac{\beta}{2} 2x_1$, $\frac{\partial p}{\partial x_2} = \frac{\beta}{2} (-2x_2)$

$$\nabla^2 p = \beta(1-1) = 0 \quad \checkmark$$

$$\vec{q} = -\frac{k}{\mu} \vec{\nabla} p = -\frac{k\beta}{2\mu} (2x_1 \hat{e}_1 - 2x_2 \hat{e}_2) = -\frac{k\beta}{\mu} (x_1 \hat{e}_1 - x_2 \hat{e}_2)$$



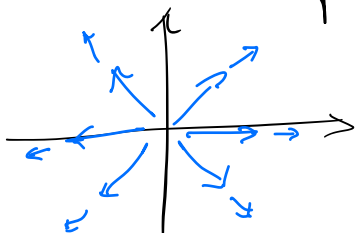
4. $p = \beta x_1 x_2 \Rightarrow \vec{q} = -\frac{k}{\mu} \vec{\nabla} p = -\frac{k\beta}{\mu} (x_2 \hat{e}_1 + x_1 \hat{e}_2)$
Looks like #3, rotated 45°

Darcy's law is linear, so superposition holds: sums of solutions are also solutions. All ^{other} 2D solutions are superpositions of these. If you can combine known solutions in a way that satisfies the given bdy cond, you've found the solution.

Bonus solution: source/sink.

Instead of $\nabla^2 p = 0$, solve $\nabla^2 p = \delta(\vec{x})$ ^{Dirac}

Sol'n is $p = p_0 \ln \frac{r}{r_0}$



Source if $p_0 > 0$
Sink if $p_0 < 0$.

^{reference radius (const)}

Darcy flow through a porous medium: $\vec{q} = -\frac{k}{\mu} \nabla p$

Empirical. But how to determine permeability k ?

Could do it in experiments, eg get a representative sample of material, impose Δp , measure $\vec{q}$. Could do simulations if you have segmented, hi-res, 3D images of a sample. Can make some analytic progress...

Kozeny-Carmen eqn says that a disordered 3D arrangement of convex, rigid, impermeable particles has permeability

$$k = \frac{\Phi_s^2}{150} \frac{\phi^3}{(1-\phi^2)}$$

Φ_s ← effective diameter
 ϕ ← porosity

$$\Phi_s = \frac{\pi^{1/3} (6V)^{2/3}}{A}$$

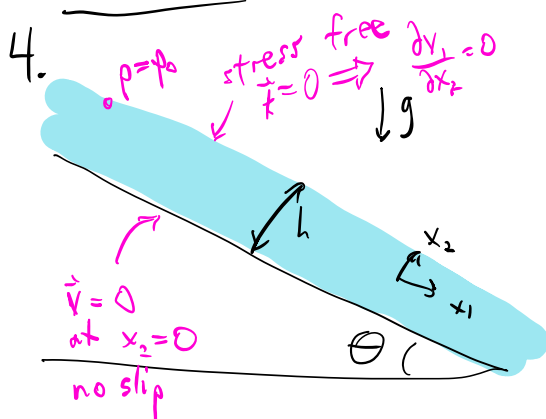
Φ_s ← sphericity
 V ← particle volume
 A ← particle surface area

For a sphere, $\Phi_s = \frac{\pi^{1/3} (6 \cdot \frac{4}{3} \pi R^3)^{2/3}}{4 \pi R^2} = 1$

$\Phi_s < 1$ for other shapes.

Assumes all particles are identical.

PS 05



Momentum equation is $\vec{a} = \frac{\Sigma \vec{F}}{m}$.

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \vec{v} + \vec{g}$$

Remember to include all relevant forces!

6. Given: Boussinesq eq's

mass: $\frac{\partial u_j}{\partial x_j} = 0$

Parameters: $U, L, T_0, \rho_0, \rho_0, \nu, \frac{k}{c_p}$ units m, s, kg, K
 Expect 3 dimless params.

Here, reference pressure p_0 was given, so assume $p = p_0 \tilde{p}$
 (In other problems, often assume $p = \rho U^2 \tilde{p}$) ↑ dimensionless pressure
 (For slow flows, often assume $p = \frac{\mu L}{U} \tilde{p}$)

Also $\vec{u} = U \tilde{u}$, $\frac{\partial}{\partial t} = \frac{U}{L} \frac{\partial}{\partial \tilde{t}}$, $\frac{\partial}{\partial x_i} = \frac{1}{L} \frac{\partial}{\partial \tilde{x}_i}$, ...

Then do algebraic substitution to change variables.

For the final

Solids, esp elastic solids

$$T_{ij} = \lambda \delta_{ij} E_{kk} + 2\mu E_{ij}$$

↑ 1st Lamé coefficient
↑ shear modulus G

Stress is linearly related to strain.

Other mat'l properties: E_Y (Young's), K (bulk), ν (Poisson's)

Waves (transverse AKA shear + longitudinal AKA pressure)

Wave reflection (stress-free boundary)

Fluids, esp incompressible Newtonian

Shear stress is proportional to shear strain rate.

$$T_{ij} = -p \delta_{ij} + \lambda D_{kk} \delta_{ij} + 2\mu \Delta_{ij}$$

↑

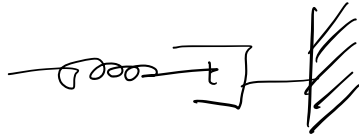
$\lambda + \frac{2}{3}\mu$ is bulk viscosity

dynamic shear viscosity

N-S & mass eq'n.

Viscoelasticity

Maxwell fluid



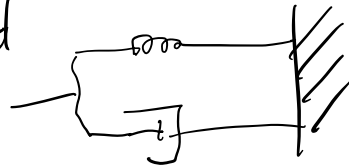
$$T_{ij} = -p \delta_{ij} + S_{ij}$$

with $S_{ij} +$

$$\lambda \frac{\partial S_{ij}}{\partial t} = 2\mu D_{ij}$$

relaxation time (pointing to λ)
shear modulus (pointing to μ)

Kelvin-Voigt solid



$$S_{ij} = G' E_{ij} + 2\mu D_{ij}$$

$$G = G' + iG''$$