Statistical-noise properties of an optical amplifier utilizing two-beam coupling in atomic-potassium vapor

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We have measured the gain and statistical-noise properties of a weak probe beam amplified through two-beam coupling in atomic-potassium vapor for both the Rabi and stimulated Rayleigh gain features. The probe-beam gain was observed to be as large as 125 for the Rabi feature, whereas the gain was typically less than 2 for the Rayleigh feature. The rms noise at the Rabi gain feature was approximately three times greater than the ideal amplifier quantum-noise limit. For the Rayleigh gain feature, the rms noise was 12 to 15 times greater than the ideal amplifier quantum-noise limit. We present a fully quantum-mechanical theory of two-beam coupling in a system of two-level atoms which includes the effects of atomic motion and pump-beam absorption. The predictions of this theory are in good agreement with the experimental data.

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I. INTRODUCTION

While the semiclassical theory of the interaction of matter with an electromagnetic field is often sufficient, a fully quantum-mechanical treatment is usually required when the statistical properties of the field are of interest [1,2]. Unlike semiclassical theory, which can readily explain effects such as photon bunching [3], a fully quantum-mechanical theory is able to explain a whole other class of nonclassical effects such as photon antibunching [4], sub-Poissonian photon statistics [5], and the quantum-noise properties of optical amplifiers and attenuators [6]. A detailed understanding of the quantum noise of the field after it interacts with matter is important, since the resulting statistical properties of the field will limit the accuracy of optical measurements.

It has been shown that even an ideal (phase-preserving) optical amplifier degrades the signal-to-noise ratio of the output field by at least a factor of $\sqrt{2}$ relative to that of the input [6,7]. In addition, it has been shown in the literature that nonclassical features (like squeezing) of a field are lost when the intensity of the field is amplified (with a phase-insensitive amplifier) by more than a factor of 2 [8]. Since the two-beam coupling gain can be as large as 100, the statistical-noise properties of the transmitted probe beam are expected to be classical. This conclusion is consistent with the predictions of the fully quantum-mechanical theory of two-beam coupling in a homogeneously broadened system of two-level atoms [9,10]. For the case of gain through the Rabi feature, the theory predicts that the amplifier can operate at the ideal amplifier quantum-noise limit when the atomic system is radiatively broadened. The minimum noise figure for the Rayleigh gain feature is equal to 4, and occurs when the atomic system is predominantly collisionally broadened. The experimental measurements of the noise properties of a probe beam amplified through two-beam coupling in atomic vapors, however, cannot be adequately explained by this theory without modifying it to include the effects of atomic motion.

In this paper, a fully quantum-mechanical theory of two-beam coupling, including the effects of atomic motion, is presented [11]. This theory is used to calculate the noise factors needed to determine the quantum-mechanical noise properties of the transmitted probe beam. The predictions of the theory are compared with the experimental measurements of the gain and noise of the transmitted probe beam.

II. EXPERIMENTAL RESULTS

The experimental setup used to measure the statistical-noise properties of the transmitted probe beam is shown in Fig. 1. An argon-ion laser was used to pump two Coherent 699-21 continuous-wave frequency-stabilized ring-dye lasers, both linearly polarized (out of the page) and operating at a wavelength of approximately 767 nm ($4\ ^2S_{1/2} \leftrightarrow 4^2P_{3/2}$ atomic transition of potassium). The pump beam was focused into a 5-mm-long potassium vapor cell with a 500-mm lens. The weak-probe beam was focused into the vapor cell with a 400-mm lens and was made to overlap the pump beam in the cell. The angle between the wave vectors of the pump and probe beams was approximately 2.5°. The transmitted probe beam was collected and directed onto detector 3, where the signal was amplified and analyzed with an rf-spectrum analyzer. The reflection off of the window of detector 3 was used to measure the gain of the transmitted probe beam with detector 1. The gain and noise properties of the transmitted probe beam were measured as a function of probe detuning from the atomic resonance frequency.

Figure 2 shows a typical plot of the gain and normalized rms noise of the transmitted probe (for a spectrum analyzer frequency of 10 MHz) as a function of probe detuning for the case in which no buffer gas was present in the potassium

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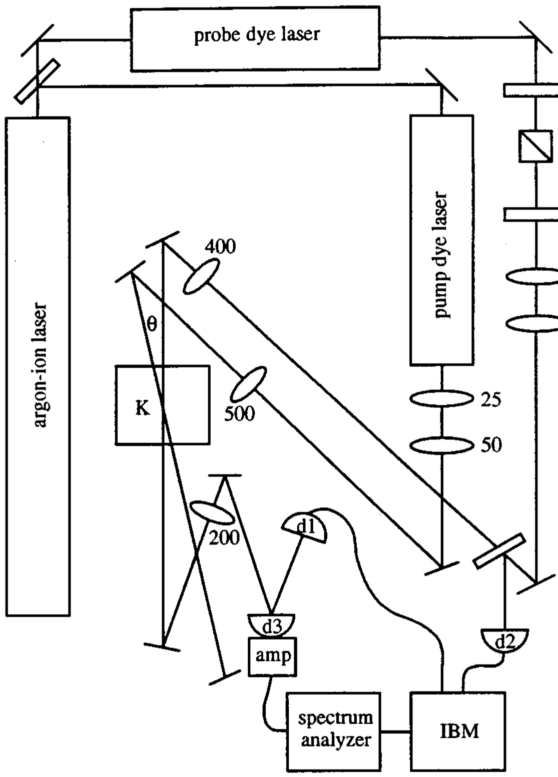

FIG. 1. Experimental setup used to measure the gain and quantum-noise properties of the transmitted probe beam.

vapor cell. Gains as large as a factor of 125 were observed under somewhat different conditions. The power of the pump beam was 140 mW and the atomic number density of the potassium in the vapor cell was approximately 5×10^{13} atoms/cm^3. The predictions for an ideal optical amplifier (see Sec. III C) are also shown on the graph. The rms noise is

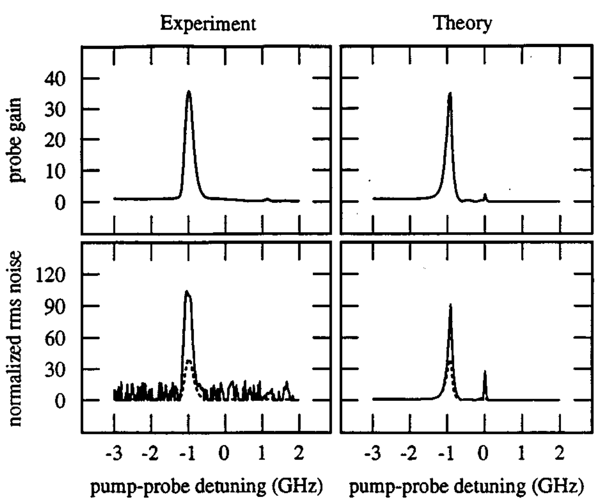

FIG. 2. Experimental measurement and corresponding theoretical predictions of the gain and rms noise of the transmitted probe beam as a function of probe detuning for the case in which no buffer gas was present in the potassium cell. The rms noise is normalized to the ideal amplifier quantum-noise limit (shown by a dashed line) in the wings.

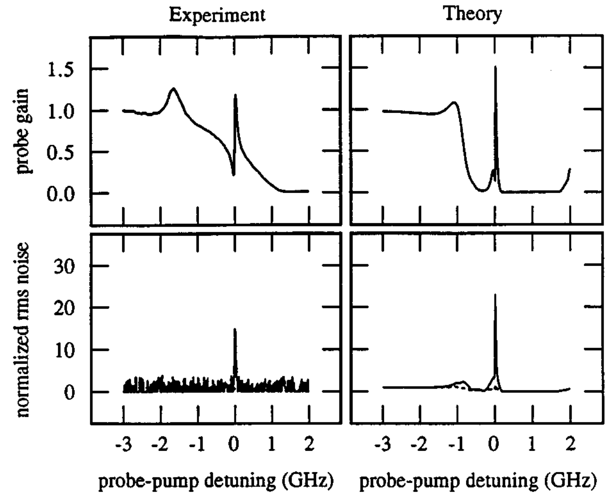

FIG. 3. Experimental measurement and the corresponding theoretical predictions of the gain and rms noise of the transmitted probe beam as a function of probe detuning for the case in which approximately 5 Torr of helium buffer gas was present in the potassium cell. The rms noise is normalized to the ideal amplifier quantum-noise limit (shown by a dashed line) in the wings.

normalized to the ideal amplifier quantum-noise limit when the frequency of the probe laser is tuned far off resonance (i.e., the gain $|g|=1$). The rms noise of the Rabi feature is typically 2.5–3.5 times greater than the ideal amplifier quantum-noise limit.

Figure 3 shows a typical plot of the gain and normalized rms noise of the transmitted probe (for a spectrum analyzer frequency of 10 MHz) as a function of probe detuning for the case in which approximately 5 Torr of helium buffer gas was present in the potassium vapor cell. The power of the pump beam was 130 mW, and the atomic number density of the potassium in the vapor cell was approximately 5×10^{13} atoms/cm^3. The rms noise of the Rayleigh feature is typically 12–15 times greater than the ideal amplifier quantum-noise limit.

The ratio of the rms noise to the ideal amplifier quantum-noise limit at the peak of the Rabi gain feature is plotted as a function of potassium number density in Fig. 4. The ratio increases slightly as the number density increases. This observation is consistent with the noise predictions of Gaeta, Boyd, and Agarwal [9] for two-beam coupling in a homogeneously broadened system of two-level atoms.

III. QUANTUM THEORY OF TWO-BEAM COUPLING

The quantum theory of two-beam coupling in a two-level atomic system is presented in this section. From this theory, the Langevin operator for the interaction can be determined. In addition, this theory can be used to predict the quantum-noise properties of a beam of light amplified (or attenuated) through two-beam coupling in an atomic vapor. The predictions of this theory are compared with the experimental results.

A. Derivation of the Langevin equation

In the quantum theory of two-beam coupling [9], the interaction between the two-level atoms (with an energy level

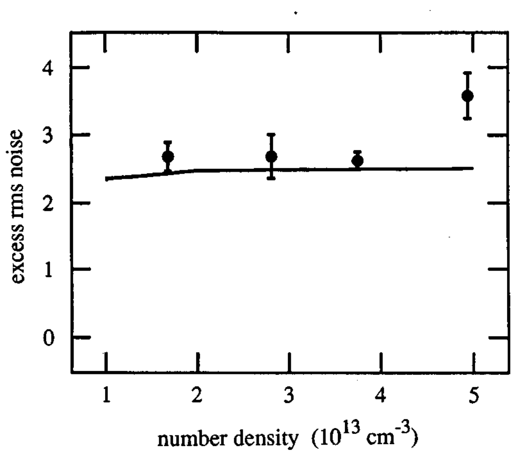

FIG. 4. Ratio of the rms noise to the ideal amplifier limit at the peak of the Rabi gain plotted as a function of potassium number density. The line shows the predictions of the theory.

separation $\hbar\omega_{ba}$) and the fields are treated in the electric-dipole approximation. All fields are assumed to propagate in the positive x direction and are linearly polarized in the z direction. The interaction Hamiltonian $\hat{V}$ is given by

$$\hat{V}=-\int \hat{\vec{P}}(\vec{r})\cdot\vec{E}_0(\vec{r},t)d^3r-\int \hat{\vec{P}}(\vec{r})\cdot\hat{\vec{E}}_1(\vec{r},t)d^3r, \quad (1a)$$

where the pump field $\vec{E}_0$ at a frequency $\omega_0=\omega_{ba}+\Delta$ given by

$$\vec{E}_0(\vec{r},t)=\hat{z}E_0e^{ik_0x-i\omega_0t}+\text{c.c.} \quad (1b)$$

is treated classically, and the probe field $\hat{\vec{E}}_1$ given by

$$\hat{\vec{E}}_1(\vec{r},t)=\hat{z}(L_xL_yL_z)^{-1/2}[\beta\hat{a}e^{ik_1x-i\omega_1t}+\text{h.a.}] \quad (1c)$$

is quantized in the volume $L_xL_yL_z$. The probe field oscillates at the frequency $\omega_1=\omega_0+\delta$ and is associated with an annihilation operator $\hat{a}$. The normalization constant β is equal to $\beta=i\sqrt{2\pi\hbar\omega_1}$. The polarization operator $\hat{\vec{P}}(\vec{r})$ is related to the dipole moment operator $\hat{\vec{\mu}}^{(q)}$ for an atom at the position $\vec{R}^{(q)}$ through the expression

$$\hat{\vec{P}}(\vec{r})=\sum_q \delta(\vec{r}-\vec{R}^{(q)})\hat{\vec{\mu}}^{(q)}, \quad (2)$$

with the summation extending over all of the atoms in the interaction region.

The total Hamiltonian $\hat{H}$ is given by $\hat{H}=\hat{H}_A+\hat{H}_F+\hat{V}$, where $\hat{H}_A$ and $\hat{H}_F$ are the unperturbed Hamiltonian for the atomic system and the field, respectively. The equation of motion for the total density operator $\hat{\rho}$ is determined from the Heisenberg equation

$$\frac{\partial\hat{\rho}}{\partial t}=-\frac{i}{\hbar}[\hat{H},\hat{\rho}]+\left(\frac{\partial\hat{\rho}}{\partial t}\right)_{\text{relax}}, \quad (3)$$

where the effects of atomic relaxation are contained in the term $(\partial\hat{\rho}/\partial t)_{\text{relax}}$. The equation of motion for the field density operator $\hat{\rho}_F$ is determined from the total density operator by tracing $\hat{\rho}$ over the atomic variables, that is, $\hat{\rho}_F=\text{Tr}_A\hat{\rho}$. The details of this calculation are given in Refs. [9] and [10] and are summarized in the Appendix. Now that the equations for the expectation value of the annihilation operator, its adjoint, and the photon number operator have been determined, the Langevin equation for the annihilation operator can be determined from the Einstein relation. The resulting Langevin equation is

$$\frac{d\hat{a}}{dt}=-\frac{\omega_1}{\omega_{ba}}\left(\frac{\alpha_0}{2|\mu|^2T_2}\right)C^{+-}(i\delta)\hat{a}+\hat{f}(t). \quad (4)$$

The properties of the Langevin operator $\hat{f}(t)$ are

$$\langle\hat{f}(t)\rangle=\langle\hat{f}^\dagger(t)\rangle=0, \quad (5a)$$

$$\langle\hat{f}^\dagger(t_1)\hat{f}(t_2)\rangle=2D_{+-}\delta(t_1-t_2), \quad (5b)$$

and

$$\langle\hat{f}(t_1)\hat{f}^\dagger(t_2)\rangle=2D_{-+}\delta(t_1-t_2), \quad (5c)$$

where

$$2D_{+-}=\frac{\omega_1}{\omega_{ba}}\left(\frac{\alpha_0}{2|\mu|^2T_2}\right)\text{Re}[Q^{+-}(i\delta)-C^{+-}(i\delta)], \quad (5d)$$

$$2D_{-+}=\frac{\omega_1}{\omega_{ba}}\left(\frac{\alpha_0}{2|\mu|^2T_2}\right)\text{Re}[Q^{+-}(i\delta)+C^{+-}(i\delta)], \quad (5e)$$

and $\alpha_0=4\pi N\omega_{ba}|\mu|^2T_2/\hbar c$ is the weak-field line-center intensity absorption coefficient.

For the case when the probe field propagates a distance l through the interaction region, the equation of motion for its annihilation operator [determined from Eq. (4)] can be replaced with an equation describing its spatial evolution by setting the time t equal to n_1x/c, where n_1 is the index of refraction of the probe field. Furthermore, the correlation functions $C^{+-}(i\delta)$ and $Q^{+-}(i\delta)$ will in general be functions of position. That is, the operating conditions of the reservoir change with position. In the case of two-beam coupling, the spatial variation of the correlation functions is a direct consequence of pump-beam absorption. The spatial variation of the pump-beam intensity $I_0(x)$ is determined from

$$\frac{dI_0}{dx}=-\alpha^D_{\text{pump}}(\Delta)I_0, \quad (6a)$$

where $\alpha^D_{\text{pump}}(\Delta)$ is the Doppler-averaged value of the pump absorption coefficient $\alpha_{\text{pump}}(\Delta)$ given by

$$\alpha_{\text{pump}}(\Delta)=\frac{\omega_0}{n_0\omega_{ba}}\frac{\alpha_0}{[1+(\Delta T_2)^2+I_0/I^0_s]}. \quad (6b)$$

The Rabi frequency Ω_0 is related to the pump intensity through the expression $I_0/I_s^0=|\Omega_0|^2T_1T_2$, where I_s^0 is the line-center saturation intensity.

Allowing for pump-beam absorption, the equation for the spatial variation of the probe-field annihilation operator is given by

$$\frac{d\hat{a}}{dx}=-\alpha(x)\hat{a}+\hat{f}(x), \tag{7a}$$

where the quantity

$$\alpha(x)\equiv\frac{\omega_1}{\omega_{ba}}\left(\frac{\alpha_0}{2|\mu|^2T_2}\right)C^{+-}(i\delta,x) \tag{7b}$$

can be shown to be identical to the modified probe-beam absorption coefficient derived from the semiclassical theory of two-beam coupling. The solution to Eq. (7a) can be easily determined by using an integrating factor, and the result is

$$\hat{a}(l)=g\hat{a}(0)+\hat{L}, \tag{8a}$$

where $\hat{a}(0)$ is the annihilation operator of the probe field before entering the interaction region, $\hat{a}(l)$ is the annihilation operator after exiting the interaction region, g given by

$$g=\exp\left[-\int_0^l dx'\,\alpha(x')\right] \tag{8b}$$

is the gain (or loss) experienced by the probe-field amplitude, and $\hat{L}$ given by

$$\hat{L}=g\int_0^l dx'\,\hat{f}(x')\exp\left[\int_0^{x'}dx''\,\alpha(x'')\right] \tag{8c}$$

is the Langevin (or noise) operator associated with the two-beam coupling process.

It can be shown that the Langevin operator $\hat{L}$ satisfies the condition

$$\langle[\hat{L},\hat{L}^\dagger]\rangle=1-|g|^2 \tag{9}$$

for an arbitrary state. Thus the commutator itself is equal to $[\hat{L},\hat{L}^\dagger]=1-|g|^2$. Note that this condition is identical to the condition derived by requiring that the output operator $\hat{a}(l)$ obey the Bosonic commutator relation. Furthermore, the expectation values $\langle\hat{L}^\dagger\hat{L}\rangle$ and $\langle\hat{L}\hat{L}^\dagger\rangle$ can be derived from the general expression for $\hat{L}$ [Eq. (8c)] and its commutation relation. Consider the expectation $\langle\hat{L}^\dagger\hat{L}\rangle$ given by

$$\langle\hat{L}^\dagger\hat{L}\rangle=|g|^2\int_0^l dx\int_0^l d\nu\langle\hat{f}^\dagger(x)\hat{f}(\nu)\rangle\exp\left[\int_0^x dx'\,\alpha^*(x')\right]\times\exp\left[\int_0^\nu d\nu'\,\alpha(\nu')\right]. \tag{10}$$

This expectation value will be of great importance in the following since it is the quantity needed to calculate the noise properties of the photocurrent. Equation (10) can be further simplified by using Eqs. (5b) and (5d) and the identity

$$\frac{d}{dx}\left\{\exp\left[\int_0^x dx'[\alpha(x')+\alpha^*(x')]\right]\right\}=[\alpha(x)+\alpha^*(x)]\exp\left[\int_0^x dx'[\alpha(x')+\alpha^*(x')]\right]. \tag{11}$$

The resulting expression for this expectation value is

$$\langle\hat{L}^\dagger\hat{L}\rangle=-\frac{1}{2}(1-|g|^2)+\frac{\omega_1}{\omega_{ba}}\left(\frac{\alpha_0}{2|\mu|^2T_2}\right)|g|^2\times\int_0^l dx\ \mathrm{Re}(Q^{+-}(i\delta,x))\times\exp\left[\int_0^x dx'[\alpha(x')+\alpha^*(x')]\right]. \tag{12}$$

For an ideal optical amplifier (with $|g|^2>1$), it can be shown that $\langle\hat{L}^\dagger\hat{L}\rangle$ is equal to $(|g|^2-1)$. Furthermore, under ideal conditions $\langle\hat{L}^\dagger\hat{L}\rangle$ is equal to zero when $|g|^2\leqslant 1$.

B. Effects of atomic motion

Atomic motion affects the two-beam coupling process in two ways: through Doppler shifts, and grating washout effects. Consider first the affects of atomic motion on the population gratings. Note that the pump field (with a frequency ω_0 and wave vector $\vec{k}_0$) and the probe field (with a frequency ω_1 and wave vector $\vec{k}_1$) set up a grating in the medium through the interference term $\exp[i(\vec{k}_1-\vec{k}_0)\cdot\vec{r}-i\delta t]$. The pump field can then scatter off of this grating, producing radiation with a frequency ω_1 and wave vector $\vec{k}_1$. That is, energy from the pump field is coherently added to the probe field. Atomic motion will cause these gratings to disperse or wash out, thus reducing the energy transfer efficiency of the pump beam into the probe beam. Grating washout processes are included phenomenologically for each atomic velocity group $\vec{v}$. As mentioned above, atomic motion also produces Doppler shifts in the frequency of the pump and probe. This effect is accounted for by performing Doppler averages.

Thus a proper description of the process of two-beam coupling in atomic vapors requires the inclusion of atomic motion. In our theory, the effects of atomic motion are introduced by first multiplying grating terms by an efficiency factor $S(\vec{v})$ and then performing a Doppler average. The grating efficiency factor $S(\vec{v})$ is close to unity when an atom moves only a small fraction of the grating period in a time T_1 before making a transition back to the ground state. However, $S(\vec{v})$ will be approximately zero if an atom moves a distance comparable to the grating period in a time T_1. A convenient choice for the grating efficiency factor is

$$S(\vec{v})=\begin{cases}\frac{1}{2}[1+\cos X(\vec{v})] & \text{for } 0<X(\vec{v})<\pi\\ 0 & \text{for } X(\vec{v})\geqslant\pi,\end{cases} \tag{13a}$$

where

$$X(\vec{v})=\frac{1}{2\varepsilon}|\vec{k}_g\cdot\vec{v}|T_1, \tag{13b}$$

$\vec{k}_g=\vec{k}_1-\vec{k}_0$ is the grating wave vector, and ε is a grating parameter which has a value greater than zero. When ε is equal to unity, the grating will be completely washed out for atoms which have moved a distance greater than one grating period in a time T_1. The parameter ε serves as a free parameter when comparing the experimental results and the theoretical predictions.

As will be shown below, the gratings terms are contained in the correlation functions $C^{+-}(i\delta)$ and $Q^{+-}(i\delta)$. The population grating terms can be isolated by examining the Bloch equations for the atoms in the presence of both the pump and probe fields. In a frame rotating with the angular frequency of the pump [see Eq. (A3b)], the Bloch equations can be written in the form

$$\begin{aligned}\frac{\partial\Phi}{\partial t}=&(\mathbf{M}_0+\Omega_0\mathbf{M}_++\Omega_0^*\mathbf{M}_-)\mathbf{\Phi}+\Psi\\&+\Omega_1e^{i(\vec{k}_1-\vec{k}_0)\cdot\vec{r}-i(\omega_1-\omega_0)t}\mathbf{M}_+\mathbf{\Phi}\\&+\Omega_1^*e^{-i(\vec{k}_1-\vec{k}_0)\cdot\vec{r}+i(\omega_1-\omega_0)t}\mathbf{M}_-\mathbf{\Phi},\end{aligned} \tag{14}$$

where the expressions for the matrices $\mathbf{M}_0$ and $\mathbf{M}_\pm$ are easily obtained from Eq. (A3d), and $\Omega_1\equiv2\mu E_1/\hbar$ is the Rabi frequency associated with the probe field which is assumed to be much smaller in magnitude than Ω_0. The dipole moment at the probe frequency ω_1 determines the two-beam coupling efficiency. Since the calculation is performed in a rotating frame, the dipole moment and inversion at the frequency ω_1 are determined from the quantity $\mathbf{\Phi}(\delta)$. Solving Eq. (14) to lowest order in Ω_1, the part of the Bloch vector oscillating at the probe-pump detuning δ is equal to

$$\Phi(\delta)=-(i\delta\mathbf{I}+\mathbf{M})^{-1}\Omega_1\mathbf{M}_+\mathbf{\Phi}^{(0)}e^{i(\vec{k}_1-\vec{k}_0)\cdot\vec{r}}, \tag{15}$$

where $\mathbf{\Phi}^{(0)}$ [given by Eq. (A5)] is the steady-state value of the Bloch vector for the atom in the presence of the pump only, and $\mathbf{M}=\mathbf{M}_0+\Omega_0\mathbf{M}_++\Omega_0^*\mathbf{M}_-$.

The grating terms in Eq. (15) can be isolated by examining the expanded form of Eq. (14) for Φ_2. The equation for Φ_2 becomes

$$\begin{aligned}\frac{\partial\Phi_2}{\partial t}=&\left(-\frac{1}{T_2}+i\Delta\right)\Phi_2-i\Omega_0\Phi_3\\&-i\Omega_1e^{i(\vec{k}_1-\vec{k}_0)\cdot\vec{r}-i(\omega_1-\omega_0)t}\Phi_3.\end{aligned} \tag{16}$$

Note further that for a weak probe field, Φ_3 can be expanded in the form

$$\Phi_3=\Phi_3^{(0)}+(\Phi_3(\delta)e^{-i(\omega_1-\omega_0)t}+\text{c.c.}), \tag{17}$$

where $\Phi_3^{(0)}$ is given by Eq. (A7c), and $\Phi_3(\delta)$ is at least of the order Ω_1 as can be seen from Eq. (15). The quantity $\Phi_3(\delta)$ is clearly the grating term in the medium. Thus the solution to Eq. (16) to lowest order in Ω_1 is clearly given by

$$\Phi_2(\delta)=\Phi_2^{(\mathrm{NG})}(\delta)+\Phi_2^{(G)}(\delta), \tag{18a}$$

where

$$\Phi_2^{(\mathrm{NG})}(\delta)=\left(\frac{1}{T_2}-i\delta-i\Delta\right)^{-1}(-i\Omega_1)\Phi_3^{(0)}e^{i(\vec{k}_1-\vec{k}_0)\cdot\vec{r}} \tag{18b}$$

and

$$\Phi_2^{(G)}(\delta)=\left(\frac{1}{T_2}-i\delta-i\Delta\right)^{-1}(-i\Omega_0)\Phi_3(\delta) \tag{18c}$$

are the nongrating and grating terms, respectively. These quantities can be rewritten in the form of Eq. (15) through use of

$$(i\delta\mathbf{I}+\mathbf{M}_0)_{2j}^{-1}=\left(i\delta+i\Delta-\frac{1}{T_2}\right)^{-1}\delta_{2j}, \tag{19a}$$

$$\begin{aligned}(i\delta\mathbf{I}+\mathbf{M})^{-1}=&(i\delta\mathbf{I}+\mathbf{M}_0)^{-1}-(i\delta\mathbf{I}+\mathbf{M}_0)^{-1}\\&\times(\Omega_0\mathbf{M}_++\Omega_0^*\mathbf{M}_-)(i\delta\mathbf{I}+\mathbf{M})^{-1},\end{aligned} \tag{19b}$$

$$\Phi_2^{(\mathrm{NG})}(\delta)=-\{(i\delta\mathbf{I}+\mathbf{M}_0)^{-1}\Omega_1\mathbf{M}_+\mathbf{\Phi}^{(0)}\}_2e^{i(\vec{k}_1-\vec{k}_0)\cdot\vec{r}}, \tag{19c}$$

$$\begin{aligned}\Phi_2^{(G)}(\delta)=&-\{[(i\delta\mathbf{I}+\mathbf{M})^{-1}-(i\delta\mathbf{I}+\mathbf{M}_0)^{-1}]\Omega_1\mathbf{M}_+\mathbf{\Phi}^{(0)}\}_2\\&\times e^{i(\vec{k}_1^r-\vec{k}_0^r)\cdot\vec{r}^r}.\end{aligned} \tag{19d}$$

Comparing Eqs. (19) with Eq. (15), the following procedure can be used to isolate the nongrating and grating parts. The nongrating terms are obtained by replacing $\mathbf{M}$ by $\mathbf{M}_0$ in the dynamical matrix $\mathbf{U}$. That is, the matrix $\mathbf{U}$ [Eq. (A6)] is replaced by the matrix $\mathbf{V}$ where $\mathbf{V}=\mathbf{U}|_{\Omega_0=0}$. The grating terms are obtained by replacing the dynamical matrix $\mathbf{U}$ by $(\mathbf{U}-\mathbf{V})$. Note that the steady-state values of the Bloch vector in the presence of the pump field are still used.

As mentioned earlier, the quantity C^{+-} is related to the nonlinear susceptibility of the medium. Thus this procedure can be used to isolate the nongrating and grating terms of C^{+-}. A similar assignment applies to the quantity Q^{+-} (the noise terms). This can be understood by realizing that in the fully quantum-mechanical theory, the Bloch equations become operator Langevin equations with the probe field now being replaced by the multimode vacuum of the radiation field, i.e.,

$$\Omega_1e^{i\vec{k}_1\cdot\vec{r}-i\omega_1t}\Phi_3\to\sum_k e^{i\vec{k}\cdot\vec{r}-i\omega_kt}S^z\hat{a}_k. \tag{20}$$

Such noise terms can be handled using standard methods. The grating contributions in these noise terms would again arise from terms like $\Omega_0\Phi_3$ or Ω_0S^z in Eq. (16).

In light of the above discussion, the effects of atomic motion are included in the following way. The correlation functions $C^{+-}(i\delta)$ and $Q^{+-}(i\delta)$ are separated into a nongrating $[C_{\mathrm{NG}}^{+-}(i\delta)$ and $Q_{\mathrm{NG}}^{+-}(i\delta)]$ and a grating $[C_G^{+-}(i\delta)$ and $Q_G^{+-}(i\delta)]$ part, that is,

$$C^{+-}(i\delta)=C_{\mathrm{NG}}^{+-}(i\delta)+[C_G^{+-}(i\delta)]S(\vec{v}) \tag{21a}$$

and

$$Q^{+-}(i\delta)=Q_{\rm NG}^{+-}(i\delta)+[Q_G^{+-}(i\delta)]S(\vec{v}), \tag{21b}$$

where

$$C_{\rm NG}^{+-}(i\delta)=|\mu|^2T_2\sum_l V_{2l}(i\delta)(0,-2\Phi_3,\Phi_1)_l, \tag{21c}$$

$$Q_{\rm NG}^{+-}(i\delta)=|\mu|^2T_2\sum_l V_{2l}(i\delta)(-2\Phi_1\Phi_1,1-2\Phi_1\Phi_2,-2\Phi_1\Phi_3)_l, \tag{21d}$$

$$C_G^{+-}(i\delta)=|\mu|^2T_2\sum_l (\mathbf{U}-\mathbf{V})_{2l}(i\delta)(0,-2\Phi_3,\Phi_1)_l, \tag{21e}$$

$$Q_G^{+-}(i\delta)=|\mu|^2T_2\sum_l (\mathbf{U}-\mathbf{V})_{2l}(i\delta)(-2\Phi_1\Phi_1,1-2\Phi_1\Phi_2,-2\Phi_1\Phi_3)_l, \tag{21f}$$

and $S(\vec{v})$ is the grating efficiency factor. Finally, the Doppler-averaged correlation functions are calculated by performing a two-dimensional Doppler integration on Eqs. (21).

C. Photocurrent fluctuations

The solution for the output annihilation operator $\hat{a}(l)$ [Eq. (8a)] can now be used to calculate the quantum-noise properties of the transmitted probe beam. The expectation value of the output photon-number operator is given by

$$\langle\hat{n}(l)\rangle=\langle(g^*\hat{a}^\dagger(0)+\hat{L}^\dagger)(g\hat{a}(0)+\hat{L})\rangle. \tag{22}$$

Using the fact that the Langevin operators are Gaussian in nature, this expression can be simplified to

$$\langle\hat{n}(l)\rangle=|g|^2\langle\hat{n}(0)\rangle+\langle\hat{L}^\dagger\hat{L}\rangle, \tag{23}$$

where g and $\langle\hat{L}^\dagger\hat{L}\rangle$ are given by Eqs. (8b) and (12), respectively. Similarly, the variance of the photon number for a highly excited coherent-state input is given by

$$\langle(\Delta\hat{n}(l))^2\rangle=|g|^2\langle\hat{n}(0)\rangle[2\langle\hat{L}^\dagger\hat{L}\rangle+1]. \tag{24}$$

Then the Fano number is given by

$$Z=[2\langle\hat{L}^\dagger\hat{L}\rangle+1], \tag{25}$$

and it is by definition equal to the ratio of the power fluctuations to the shot noise. Under ideal operating conditions, the Fano number is given by

$$Z=\begin{cases}(2G-1) & \text{for } G>1 \text{ (ideal amplifier quantum-noise limit)}\\ 1 & \text{for } G\leqslant 1 \text{ (ideal attenuator quantum-noise limit)},\end{cases} \tag{26}$$

where $G=|g|^2$. In addition, the noise figure defined by $\mathscr{F}\equiv(S/N)^2_{\rm input}/(S/N)^2_{\rm output}$ is equal to

$$\mathscr{F}=\frac{Z}{G} \tag{27}$$

for the two-beam coupling process.

IV. NUMERICAL RESULTS

The theoretical predictions for the gain and noise of the transmitted probe beam were generated numerically by integrating the Doppler-averaged pump propagation equation [Eqs. (6)] and using this result to calculate the Doppler-averaged atomic-polarization correlation functions. These correlation functions were used to calculate the gain g and $\langle\hat{L}^\dagger\hat{L}\rangle$. The Fano number of the transmitted probe field was then determined from Eq. (25). Finally, the normalized rms noise was calculated by taking the square root of the product of the Fano number at the detector, and the probe transmittance through the cell.

Figure 2 shows a typical experimental measurement of the gain and normalized rms noise for the Rabi gain feature with corresponding theoretical predictions for the case in which no buffer gas present in the cell. The pump power entering the cell was equal to 140 mW. The best agreement between the predictions of the theory and the experimental results was obtained for an entering pump intensity of 115 W/cm^2, a crossing angle of 2.5°, a pump detuning of -0.7 GHz, and a grating parameter ε of $\frac{1}{3}$. The agreement between the theory and experimental results is quite good.

Figure 3 shows a typical experimental measurement of the gain and normalized rms noise for the Rayleigh gain feature with the corresponding theoretical predictions for the case in which approximately 5 Torr of helium buffer gas is present in the cell. The pump power entering the cell was equal to 130 mW. The best agreement between the predictions of the theory and the experimental results was obtained for an entering pump intensity of 80 W/cm^2, a crossing angle of 2.5°, a pump detuning of -1.5 GHz, and a grating parameter ε of $\frac{1}{3}$. The agreement between the theory and experimental results is good, but not as good as that for the Rabi feature. The increased sensitivity to the crossing angle and buffer gas pressure for the Rayleigh gain feature makes it very difficult to determine the precise experimental conditions.

The ratio of the rms noise to the ideal amplifier quantum-noise limit at the peak of the Rabi gain feature is plotted as a function of potassium number density in Fig. 4. The curve shows the prediction of the theory. This ratio increases slightly as the number density increases. The theoretical predictions display the correct qualitative behavior.

V. CONCLUSIONS

In conclusion, a theoretical and experimental investigation of the quantum-noise properties of a probe beam amplified through two-beam coupling in atomic-potassium vapor has been presented. For a spectrum analyzer frequency of 10 MHz, the rms noise at the Rabi gain feature is typically 2.5–3.5 times greater than that of the ideal amplifier quantum-noise limit. For the Rayleigh gain feature, the rms noise at is typically 12–15 times greater than that of the ideal amplifier quantum-noise limit, and decreases as the helium buffer gas pressure increases. These results are in good agreement with the predictions of a fully quantum-mechanical single-mode theory of two-beam coupling, including the effects of atomic motion and pump-beam absorption.

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APPENDIX

In this appendix we outline some of the intermediate steps in the calculation of the Langevin equation describing the quantum dynamics of the probe field. The master equation for the field density operator is given by [9] and [10],

$$\frac{\partial \hat{\rho}_F}{\partial t} = -\frac{|\beta^2|N}{2\hbar^2}(Q^{+-}(i\delta)[\hat{a}^\dagger,[\hat{a},\hat{\rho}_F]] + C^{+-}(i\delta) \times[\hat{a}^\dagger,\{\hat{a},\hat{\rho}_F\}]) + \text{H.a.}, \tag{A1}$$

where N is the atomic number density. The quantities $C^{+-}(i\delta)$ and $Q^{+-}(i\delta)$ are the Laplace transforms of specific linear combinations of two-time atomic-polarization correlation functions [9]. Physically, $C^{+-}(i\delta)$ is proportional to the nonlinear susceptibility that appears in the semiclassical theory of two-beam coupling. The quantity $Q^{+-}(i\delta)$, however, has no counterpart in semiclassical theories, and it represents quantum fluctuations of the atomic system.

For the case of a two-level atom, the correlation functions $C^{+-}(i\delta)$ and $Q^{+-}(i\delta)$ are determined from the optical Bloch equations. First, the polarization operator is rewritten in the form

$$\hat{\vec{P}}(\mathbf{r}) = \vec{\mu}\hat{S}^+\delta(\vec{r}-\vec{R}) + \text{H.a.}, \tag{A2}$$

where $\hat{S}^+$, its adjoint $\hat{S}^-$, and $\hat{S}^z = \frac{1}{2}[\hat{S}^+,\hat{S}^-]$ obey the commutation relations for a spin-$\frac{1}{2}$ system. The quantity $\vec{\mu}$ is defined by $\vec{\mu} \equiv \langle\vec{\mu}\rangle$. Then the matrix form of the Bloch equations for an atom located at the position $\vec{R}$ interacting with the pump field are given by

$$\frac{\partial \mathbf{\Phi}}{\partial t} = \mathbf{M}\mathbf{\Phi} + \mathbf{\Psi}, \tag{A3a}$$

where the components of the Bloch vector $\mathbf{\Phi}$ and the vector $\mathbf{\Psi}$ are

$$\Phi_1 = \langle S^+\rangle e^{i(\vec{k}_0\cdot\vec{R}-\omega_0 t)},\ \ \Phi_2 = \Phi_1^*,\ \ \Phi_3 = \langle S^z\rangle \tag{A3b}$$

and

$$\Psi_1 = \Psi_2 = 0,\ \ \Psi_3 = -\frac{1}{2T_1}, \tag{A3c}$$

respectively. The matrix $\mathbf{M}$ is equal to

$$\mathbf{M} = \begin{bmatrix} -1/T_2 - i\Delta & 0 & i\Omega_0^* \\ 0 & -1/T_2 + i\Delta & -i\Omega_0 \\ i\Omega_0/2 & -i\Omega_0^*/2 & -1/T_1 \end{bmatrix}, \tag{A3d}$$

where $\Delta \equiv \omega_0 - \omega_{ba}$ is the detuning of the pump laser from the atomic resonance, T_1 is the population decay time, T_2 is the dipole dephasing time, and the Rabi frequency Ω_0 is defined by $\Omega_0 \equiv 2\mu_z E_0(\vec{r})/\hbar$. The correlation functions $C^{+-}(i\delta)$ and $Q^{+-}(i\delta)$ can then be calculated in terms of the solution of Eq. (A3a) by using the quantum regression theorem. The results given in terms of the steady-state solution of the Bloch equations and a matrix $\mathbf{U}$ are [9]

$$C^{+-}(i\delta) = |\mu|^2 T_2 \sum_l U_{2l}(i\delta)(0,-2\Phi_3^{(0)},\Phi_1^{(0)})_l \tag{A4a}$$

and

$$Q^{+-}(i\delta) = |\mu|^2 T_2 \sum_l U_{2l}(i\delta)(-2\Phi_1^{(0)}\Phi_1^{(0)}, 1-2\Phi_1^{(0)}\Phi_2^{(0)}, -2\Phi_1^{(0)}\Phi_3^{(0)})_l, \tag{A4b}$$

where the definition $\mu \equiv \mu_z$ has been used to simplify the notation. The steady-state solution of the Bloch equations in the presence of the pump $\Phi^{(0)}$ and the matrix $\mathbf{U}$ are determined from

$$\mathbf{\Phi}^{(0)} = -\mathbf{M}^{-1}\mathbf{\Psi} \tag{A5}$$

and

$$T_2\mathbf{U}(i\sigma) = -(i\sigma\mathbf{I} + \mathbf{M})^{-1}, \tag{A6}$$

where $\mathbf{I}$ is the identity matrix. The results of the calculations are

$$\Phi_1^{(0)} = -\frac{\Omega_0^* T_2(\Delta T_2 + i)}{2P(0)}, \tag{A7a}$$

$$\Phi_2^{(0)} = (\Phi_1^{(0)})^* = -\frac{\Omega_0 T_2(\Delta T_2 - i)}{2P(0)}, \tag{A7b}$$

$$\Phi_3^{(0)} = -\frac{(1+(\Delta T_2)^2)}{2P(0)} \tag{A7c}$$

and

$$U_{21}(i\sigma)=\frac{\Omega_0^2 T_1 T_2}{2P(\sigma)}, \tag{A7d}$$

$$U_{22}(i\sigma)=-\frac{2(\gamma\sigma T_2+i)[(\sigma-\Delta)T_2+i]-|\Omega_0|^2 T_1 T_2}{2P(\sigma)}, \tag{A7e}$$

$$U_{23}(i\sigma)=-\frac{\Omega_0 T_1[(\sigma-\Delta)T_2+i]}{P(\sigma)}, \tag{A7f}$$

where

$$P(\sigma)=(1-i\gamma\sigma T_2)[(1-i\sigma T_2)^2+(\Delta T_2)^2] + (1-i\sigma T_2)|\Omega_0|^2 T_1 T_2 \tag{A7g}$$

and $\gamma \equiv T_1/T_2$.

The master equation [Eq. (A1)] can be converted into an equation for the temporal evolution of the expectation value of the moments of the probe field operator. In the Schrödinger picture, it can be shown that the derivative of the expectation value of a field operator $\hat{G}$ is given by [12]

$$\frac{d}{dt}\langle\hat{G}\rangle=\frac{d}{dt}\mathrm{Tr}(\hat{G}\hat{\rho}_F)=\mathrm{Tr}\left(\hat{G}\frac{\partial\hat{\rho}_F}{\partial t}\right). \tag{A8}$$

Furthermore, with the help of the identities

$$\mathrm{Tr}(\hat{G}[\hat{A},[\hat{B},\hat{\rho}_F]])=\langle[\hat{B},[\hat{A},\hat{G}]]\rangle \tag{A9a}$$

and

$$\mathrm{Tr}(\hat{G}[\hat{A},\{\hat{B},\hat{\rho}_F\}])=\langle\{\hat{B},[\hat{G},\hat{A}]\}\rangle, \tag{A9b}$$

the evolution of the expectation value of a field operator $\hat{G}$ is determined from the equation

$$\frac{d}{dt}\langle\hat{G}\rangle=-\frac{|\beta|^2 N}{2\hbar^2}\{Q^{+-}(i\delta)\langle[\hat{a},[\hat{a}^\dagger,\hat{G}]]\rangle+C^{+-}(i\delta) \times\langle\{\hat{a},[\hat{G},\hat{a}^\dagger]\}\rangle+(Q^{+-}(i\delta))^*\langle[\hat{a}^\dagger,[\hat{a},\hat{G}]]\rangle -(C^{+-}(i\delta))^*\langle\{\hat{a}^\dagger,[\hat{G},\hat{a}]\}\rangle\}. \tag{A10}$$

The equations of motion for the expectation value of the annihilation operator $\hat{a}$, its adjoint $\hat{a}^\dagger$, and the photon number operator $\hat{n}=\hat{a}^\dagger\hat{a}$ are

$$\frac{d}{dt}\langle\hat{a}\rangle=-\frac{\omega_1}{\omega_{ba}}\left(\frac{\alpha_0}{2|\mu|^2 T_2}\right)C^{+-}(i\delta)\langle\hat{a}\rangle, \tag{A11a}$$

$$\frac{d}{dt}\langle\hat{a}^\dagger\rangle=-\frac{\omega_1}{\omega_{ba}}\left(\frac{\alpha_0}{2|\mu|^2 T_2}\right)(C^{+-}(i\delta))^*\langle\hat{a}^\dagger\rangle, \tag{A11b}$$

and

$$\frac{d}{dt}\langle\hat{n}\rangle=-\frac{\omega_1}{\omega_{ba}}\left(\frac{\alpha_0}{2|\mu|^2 T_2}\right)\Big\{[C^{+-}(i\delta)+(C^{+-}(i\delta))^*]\langle\hat{n}\rangle +\frac{1}{2}[C^{+-}(i\delta)+(C^{+-}(i\delta))^*-Q^{+-}(i\delta) -(Q^{+-}(i\delta))^*]\Big\}, \tag{A11c}$$

respectively. Equations (A11) are used in Sec. III A to derive the Langevin equation (4).

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