ELECTROMAGNETIC THEORY & FOURIER OPTICS

NICHOLAS GEORGE

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Optics 462 Electromagnetic Theory and Fourier Optics
Physical Optics II

Monday/Wednesday **116 Wilmot** **Nicholas George**
9:00 AM to 10:20 AM **ngeorge@troi.cc.rochester.edu**

Course Description/Syllabus Spring 2006

Optics 462 Electromagnetic Theory and Fourier Optics

An advanced course in electromagnetic theory with selected topics in Fourier optics and statistical optics or holography. We emphasize electromagnetic wave propagation, scattering and diffraction. Lectures and problem sets will include the following subjects. Graduate standing and Optics 461 are required.

Maxwell's Equations

Review of EMT/Vector Analysis/Signal Representation

Plane Waves

Theory of Dielectrics

Wave Equation Solutions (computer oriented)

Scattering by Cylinders, Spheres, and Rough Surfaces

Energy Conservation Theorems/Uniqueness

Diffraction Theory/Fourier Optics Topics

Statistical Optics (Speckle) or Holography

Guided Waves (Fibers and Waveguides)

LECTURE NOTES Spring 2006

CONTENTS

MAXWELL'S EQUATION
- REVIEW DERIVATIONS, PHYSICAL CONCEPTS
- DISCUSS TEXTBOOKS
- MATH & VECTOR ANAL TOPICS
- ENERGY EQUATIONS - THEOREMS

BACKGROUND TOPICS NOT COVERED

FOURIER TRANSFORM, LAPLACE TRANSFORM, SIGNAL REPRESENTATIONS

HTD / FOURIER TRANSFORM WITH RESPECT TO TIME
- EQUATIONS OF MAJOR INTEREST / POYNTING II

SCALAR WAVE EQUATION
- CARTESIAN
- CYLINDRICAL
- SPHERICAL

ALL ABOUT PLANE WAVES
- NORMAL INCIDENCE, PHASORS
- SINGLE INTERFACE
- SMITH CHART
- E-WAVES, H-WAVES, FRESNEL EQUATIONS
- ORTHOGONAL EXPANSIONS

Optics 462 **Reference List** **January 2006**
Nicholas George

For the course texts by C.A. Balanis and J.D. Jackson are recommended.

*** C.A. Balanis, **Advanced Engineering Electromagnetics**, John Wiley & Sons, 1989.

David J. Griffiths, **Introduction to Electrodynamics**, Third Edition, Prentice Hall, 1999.

*** John David Jackson, **Classical Electrodynamics**, Third Edition, John Wiley & Sons, Inc., 1998.

Oleg D. Jefimenko, **Electricity and Magnetism**, Second Edition, Electret Scientific Company, 1989.

Jin Au Kong, **Electromagnetic Wave Theory**, Second Edition, John Wiley & Sons, Inc., 1990.

C.H. Papas, **Theory of Electromagnetic Wave Propagation**, McGraw-Hill Book Co., 1965.

W.R. Smythe, **Static and Dynamic Electricity**, Third Edition, Revised, Hemisphere Publishing Co., 1989.

Max Born and Emil Wolf, **Principles of Optics**, Seventh Edition, Pergamon Press, 1999.

OPTICS 462 **SPRING 2006**

NICHOLAS GEORGE, 207 WILMOT

These lecture notes closely follow the lectures being presented in the spring semester 2006. They are not meant to replace careful study of one or more of the textbooks listed for the course. They are simply presented, as is, so that the students can fill-in their own class notes. Questions and class discussion are encouraged.

Pages are also inserted to describe the subject being planned for in the coming lecture or in the next few lectures.

Students are invited to come by my office in 207 Wilmot at most times during the normal working day and particularly from 1:00 PM to 4:00 PM on Monday through Friday.

The lectures start with a discussion of

(1) Maxwell's equations in a time-dependent form using script $\mathcal{E}(r,t)$, $\mathcal{H}(r,t)$ etc. to denote the field vectors
(2) Review Signal Representations
(3) Review Transform Theory
(4) Vector Wave Equation
(5) Boundary Conditions
(6) Perfect Conductors (lots of detail)
(7) What constitutes a solution of Maxwell's Equations
(8) Simple solutions to see propagation
(9) Energy stored
(10) Poynting's Theorem (time dependent form)

OPTICS 462 LECTURE NOTES 12 JAN 2005

N. George

For the first lecture, we will start with <u>MAXWELL'S EQUATIONS</u>

1) $$\nabla \times \underline{\mathcal{E}}(\underline{r}, t) = -\frac{\partial \underline{\mathcal{B}}(\underline{r}, t)}{\partial t}$$

2) $$\nabla \times \underline{\mathcal{H}}(\underline{r}, t) = \underline{\mathcal{J}}(\underline{r}, t) + \frac{\partial \underline{\mathcal{D}}(\underline{r}, t)}{\partial t}$$

In your preparation for the course (at the level of D.J. Griffith's book - per attached list), you have had a careful derivation of these.

The first 2 weeks we will review:

Maxwell's Equations - in 'simple' medium   define 'simple'

Vector Analysis: do cover of Jackson

μ, σ, ϵ scalars
homogeneous
isotropic

Course emphasis: Propagation / Radiation Topics light / em waves

Discuss syllabus handout

GOAL: Develop ability to solve any problem in Propagation / Radiation

Spend last 15 min to overview specific topics
Use yellow summary formulas - advise for tests

TIME DEPENDENT FORM OF MAXWELL'S EQUNS Jan 2005

Consider unbounded free space or simple dielectrics

Independent Maxwell's Equations:

1) $$\nabla \times \underline{\mathcal{E}}(\underline{r},t) = -\frac{\partial \underline{\mathcal{B}}(\underline{r},t)}{\partial t}$$

2) $$\nabla \times \underline{\mathcal{H}}(\underline{r},t) = \underline{\mathcal{J}}(\underline{r},t) + \frac{\partial \underline{\mathcal{D}}(\underline{r},t)}{\partial t}$$

Give detailed derivation of (3), (4) starting with

$$0 = \nabla \cdot (\nabla \times \underline{\mathcal{E}}) = -\nabla \cdot \frac{\partial \underline{\mathcal{B}}(\underline{r},t)}{\partial t}$$

...

3) $$\nabla \cdot \underline{\mathcal{B}}(\underline{r},t) = 0$$

$$0 = \nabla \cdot \nabla \times \underline{\mathcal{H}}(\underline{r},t) = \nabla \cdot \underline{\mathcal{J}}(\underline{r},t) + \nabla \cdot \frac{\partial \underline{\mathcal{D}}(\underline{r},t)}{\partial t}$$

...

4) $$\nabla \cdot \underline{\mathcal{D}}(\underline{r},t) = \rho_{placed}$$

Equation of Continuity

$$\nabla \cdot \underline{\mathcal{J}}(\underline{r},t) = -\frac{\partial}{\partial t}\rho(\underline{r},t)$$

$$\underline{F} = q\underline{\mathcal{E}} + q\underline{v} \times \underline{\mathcal{B}}$$

Here, we emphasize the point that 1) & 2) are independent equations from which we can derive the 3) & 4) divergence conditions
&
$\therefore \rightarrow$ A solution of M.E. is considered as a solution of (1) & (2)

Jan 2005

OPTICS 462
DISCUSSION OF BOOKS

TEXTS:
JACKSON 3rd Edition:

Ch		
	Intro	1-24
6	ME/Cons	237-282
7	Plane waves	295-351
8	WG/Fiber	352-406
9	Radiating Sys	407-455
10	Scatter/Diffn	456-513

* Treatment in Jackson better
* Order of topics in Balanis closer to lecture order

17 wks x 15 = 255 pages

$$\begin{array}{r} 513 \\ 237 \\ \hline 276 \end{array} \quad \text{vs} \quad \begin{array}{r} 309 \\ 100 \\ \hline 409 \end{array}$$

200 in Jackson / 250 in Balanis

CA Balanis

1	(t), $e^{i\omega t}$ EM	1-41
2	Elec Prop of Matter	42-103
3.	Wave Equation & Soln.	104-128
4	Wave Propagation & Polarization	129-179
5	Reflection & Transmission	180-253
6	Vector Potential	254-309
7.	WaveGuide Metallic/Dielectric	352-469) X
11.	Scattering: Cylind./Sphere	570-670

OTHER TOPICS
FOURIER OPTICS
ANGULAR SPECTRUM
WHITE LIGHT
INTERFERENCE
SIGNAL REPRESENTATION

REVIEW OF DERIVATION MAXWELL'S EQUATIONS

PLAN : ELECTROSTATICS 2 PAGES

MAGNETOSTATICS
FARADAY'S INDUCTION LAW 1 PAGE

MAXWELL'S DISPLACEMENT CURRENT 1 PAGE

LIST ESSENTIAL

VECTOR ANALYSIS PHYSICS DISCOVERIES
DIFFERENTIAL EQUATION
MATH TOPICS
DELTA FUNCTIONS

M/VA

EM Theory & Helmholtz Theorem

1. $\nabla \times \underline{\mathcal{E}} = -\frac{\partial B}{\partial t}$
2. $\nabla \times \mathcal{H} = \underline{J} + \frac{\partial D}{\partial t}$
3. $\nabla \cdot \underline{\mathcal{E}} = \rho/\epsilon_0$
4. $\nabla \cdot \underline{B} = 0$

$B = \mu H$
$D = \epsilon E$

Sources are J and ρ
Give rise to fields, say, $\underline{\mathcal{E}}$ & $\underline{\mathcal{H}}$

1., 2. Independent
3., 4. Derived

Are these equations enough to determine $\underline{\mathcal{E}}, \underline{H}$?

These provide us with the $\left.\begin{array}{l}\text{divergence} \\ \text{\& the curl}\end{array}\right\}$ of $\underline{\mathcal{E}}, \underline{\mathcal{H}}$

Add Boundary Conditions that the fields go to zero at ∞ and

Helmholtz' Theorem tells us

Given $\nabla \cdot \underline{F} = D(\underline{r})$
$\nabla \times \underline{F} = C(\underline{r})$

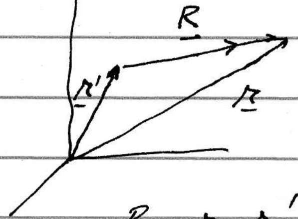
Necessary and sufficient conditions to determine $\underline{F}$, namely

$D(\underline{r})$, $C(\underline{r})$ $\rightarrow 0$ as $r \to \infty$ fast enough

$\nabla \cdot C = 0$

Griffiths App B, 555-557

$$\boxed{\underline{F}(\underline{r}) = \nabla\left(\frac{-1}{4\pi}\int \frac{\nabla' \cdot \underline{F}(\underline{r}')}{|\underline{R}|}\, dv'\right) + \nabla \times \left(\frac{1}{4\pi}\int \frac{\nabla' \times \underline{F}(\underline{r}')}{|\underline{R}|}\, dv'\right)}$$



$\underline{R} = \underline{r} - \underline{r}'$

If one does not require $\begin{array}{c}\nabla \cdot F \\ \nabla \times \underline{F}\end{array}$ to go to zero faster than $\frac{1}{r^2}$ as $r \to \infty$,

one can find a $\underline{F}(\underline{r})$ but it is not unique: $\underline{F}(\underline{r}) + \underline{P}$

add any $\underline{P}$ constant vector is still a solution.

Doing this lets us drop the $\frac{1}{r^2}$ requirement.

$$\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}$$

Outline

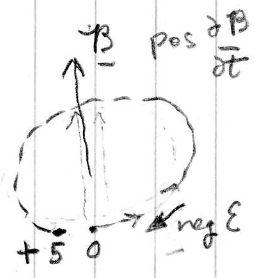
Electrostatics $\underline{E}$	**Magnetostatics** $\underline{B}$
Coulomb's Law <u>force</u> between charges	Oersted } moving charges = current 1820 } currents exert forces on each other Ampere
Electric Field at remote point due to charge or to collection of charges Gauss' Law	Magnetic field at remote point
	Continuity equation : conservation of charge
Scalar Potential : Handy descriptor for field	Faraday : field through loop is changing 1831 and this sets up an emf

$$\underbrace{\qquad\qquad\qquad}_{\text{Connects } E \,\&\, B}$$

$$\oint_{G} \underline{E}\cdot d\underline{l} = \int \nabla \times \underline{E} \cdot d\underline{a} = -\int \frac{\partial B}{\partial t} \cdot da$$

presto

$$\nabla \times E = -\frac{\partial B}{\partial t}$$



induced current

Lenz' Law

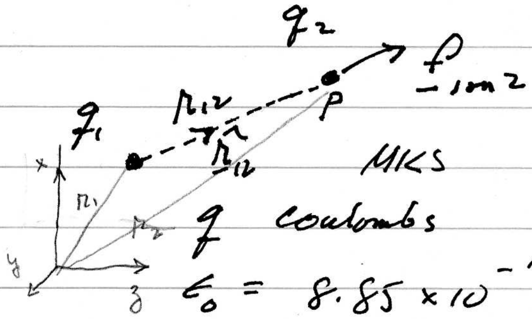
1/2

OUTLINE MAXWELL'S EQUATIONS FARADAY'S LAW

$$\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}$$

ELECTROSTATICS : COULOMB'S LAW* (Priestly's Law too) EXPERIMENTS

$$* \quad \underline{f}_{12} = \frac{q_1 q_2}{4\pi \underbrace{\epsilon_0 \epsilon_r}_{\epsilon} r_{12}^2} \hat{\underline{r}}_{12}$$



MKS

q coulombs

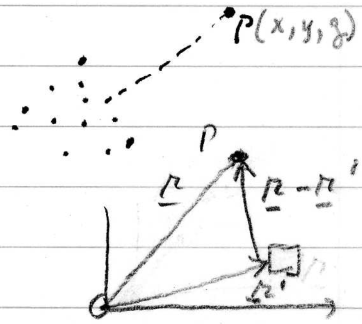
$\epsilon_0 = 8.85 \times 10^{-12}$ farad/meter

DEFINE ELECTRIC FIELD
due to q_1 @ q_2

$$\underline{E}(\underline{r}) = \frac{\underline{f}_{1\to 2}}{q_2} = \frac{q_1}{4\pi\epsilon_0 r_{12}^2} \hat{\underline{r}}_{12}$$

SUPERPOSITION : MANY CHARGES - ADDITIVE AT P

$$\underline{E}(\underline{r}) = \frac{1}{4\pi\epsilon_0} \sum_m \frac{q_m}{|\underline{r}_m - \underline{r}|^2} \frac{\underline{r}_p - \underline{r}_m}{|\underline{r}_m - \underline{r}|}$$



$$* \quad \underline{E}(\underline{r}) = \frac{1}{4\pi\epsilon_0} \int_{space} \rho(\underline{r}') \frac{\underline{r}-\underline{r}'}{|\underline{r}-\underline{r}'|^3} dv'$$

$$\rho(\underline{r}') = \frac{\text{charge}}{\text{vol}} \text{ at } \underline{r}'$$

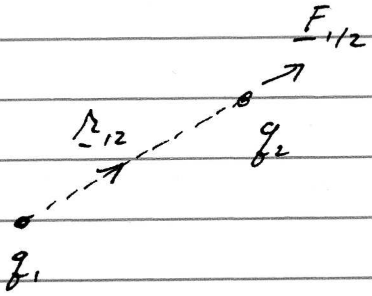
FORCES: $\underline{f}(\underline{r}) = q\left(\underline{E}(\underline{r}) + \underline{v} \times \underline{B}\right)$

MAXWELL'S EQUATIONS REVIEW - DERIVATIONS

$$\nabla \times \underline{\mathcal{E}}(\underline{r},t) = -\frac{\partial \underline{B}(\underline{r},t)}{\partial t}$$

units $\begin{cases} \text{MKS} \\ \text{SI} \end{cases}$

D. J. Griffiths



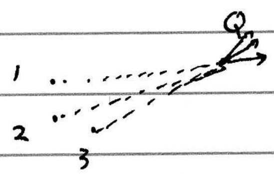
COULOMB'S LAW

Force of (1) pt charge on (2) pt charge

$$\underline{F}_{1/2} = \frac{q_1 q_2}{4\pi\epsilon_0 \, r_{12}^2} \hat{\underline{r}}_{12}$$

permitivity of free space $\epsilon_0 = 8.85 \times 10^{-12} \frac{C^2}{N\, m^2}$

Superposition:



$$\underline{F}_{on\,Q} = \underline{F}_{1q} + \underline{F}_{2q} + \underline{F}_{3q}$$

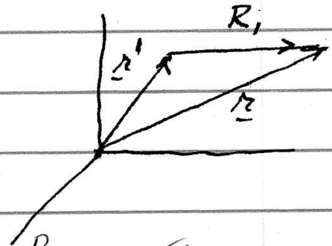
etc.

ELECTRIC FIELD : Concept, Definition

$$\underline{\mathcal{E}}_Q = \frac{q_1}{4\pi\epsilon_0 r_{1Q}^2} \hat{\underline{r}}_{1Q} + \cdots + \frac{q_n}{4\pi\epsilon_0 r_{nQ}} \hat{\underline{r}}_{nQ} = \sum_n \frac{q_{nQ}}{4\pi\epsilon_0 r_{nQ}^2} \hat{r}_{nQ}$$

....

$$\underline{\mathcal{E}}_Q = \frac{1}{4\pi\epsilon_0} \int$$



$$\underline{\mathcal{E}}(\underline{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\underline{r}')}{|\underline{r}-\underline{r}'|^2} \underbrace{\widehat{(\underline{r}-\underline{r}')}}_{= \hat{\underline{R}}_1} \, dv'$$

BASIS OF ELECTROSTATICS

GAUSS'S LAW

$$\nabla \cdot \underline{\mathcal{E}}(\underline{r}) = \frac{\rho(\underline{r})}{\epsilon_0}$$

$$\nabla \cdot \left(\frac{\hat{\underline{R}}_1}{|\underline{r}-\underline{r}'|^2} \right) = 4\pi\, \delta^3(\underline{R}_1)$$

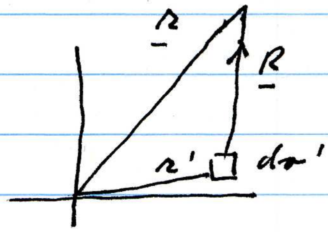
∇ operates on $\underline{r}$

prove:

$$\nabla \times \underline{\mathcal{E}}(\underline{r}) = 0 \Rightarrow \text{potential}$$

Electrostatics - Cont

Divergence & Curl of $\underline{E}$



$$\underline{E}(\underline{r}) = \frac{1}{4\pi\epsilon_0} \int_{\text{space}} \frac{\rho(\underline{r}')\,\hat{\underline{R}}}{R^2}\, dv'$$

$$\oint_S \underline{E} \cdot d\underline{a} = \frac{Q_{enclosed}}{\epsilon_0}$$

$$\nabla \cdot \underline{E} = \frac{\rho}{\epsilon_0} \rightarrow \frac{Q_{pt}}{\epsilon_0\, \delta V} = \frac{1}{\epsilon_0} Q_{pt}\, \delta(x)\delta(y)\delta(z)$$

i) Using $\int \underline{E} \cdot d\underline{a} = \int \nabla \cdot \underline{E}\, dv = \frac{1}{\epsilon_0} \int \rho'\, dv'$

ii) Direct calculation using $\nabla \left(\frac{\underline{R}}{R^2}\right) = -4\pi\, \delta(\underline{R})$

$$\nabla \times \underline{E} = 0$$

i) Single pt charge, idea of potential $E = \frac{q}{4\pi\epsilon R^2}\hat{R}$

$$\int_a^b \underline{E} \cdot d\underline{l} = \frac{q}{4\pi\epsilon}\left(\frac{-1}{R}\right)\Big|_a^b = \frac{-q}{4\pi\epsilon}\left(\frac{1}{R_b} - \frac{1}{R_a}\right) \Rightarrow a \to b \to a = 0$$

Think Helmholtz Partition theorem

ii) Direct calculation $\frac{1}{4\pi\epsilon} \int \rho(\underline{r}')\, \nabla \times \frac{\hat{R}}{R^2}\, dv' = 0$

Scalar Potential

$$\underline{E} = -\nabla V$$

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}$$

Poisson's Equation

Three Solutions: $V(\underline{r}) = \int \frac{\rho(\underline{r}')\, dv'}{4\pi\epsilon_0 r'}$

i) Superposition; ii) Assert, prove; iii) Green's Fn

Historical Pause as Magnetostatics Develops $\Rightarrow \underline{B}$

Thereafter (1831) Faraday comes along - experiments to prove:

Induced EMF $\mathcal{E} = \oint \underline{E} \cdot d\underline{l} = -\frac{d\Phi}{dt} = -\int \frac{\partial \underline{B}}{\partial t} \cdot d\underline{a}$

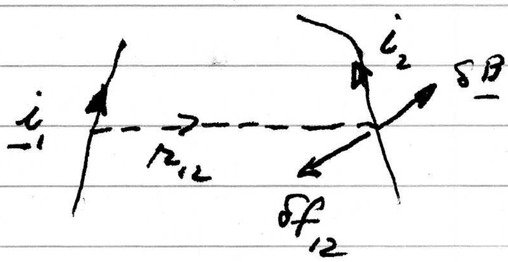
$$\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t} \quad \text{"} \int \nabla \times \underline{E} \cdot d\underline{a} = -\int \frac{\partial \underline{B}}{\partial t} \cdot d\underline{a}$$

1/2

FORCES

MAGNETOSTATICS +

WIRES CARRYING CURRENTS EXERT FORCES $\left\{ \begin{aligned} i_1\, \underline{dl}_1 &= \underline{J}\, \delta a\, \delta l_1 \\ &= \underline{J}\, dv' \end{aligned} \right.$



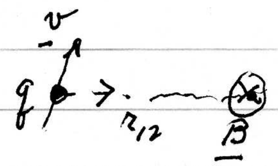
Force law 2:

$$\delta f_{1\to 2} = \frac{\mu_0}{4\pi}\, \underline{i}_2\, \underline{dl}_2 \times \left(\frac{\underline{i}_1 \times \hat{\underline{r}}_{12}}{r_{12}^2}\, dl_1 \right)$$

$$\frac{\mu_0}{4\pi}\, i_1 i_2\, \frac{\underline{dl}_2 \times (\underline{dl}_1 \times \hat{\underline{r}}_{12})}{r_{12}^2}$$

MOVING CHARGES

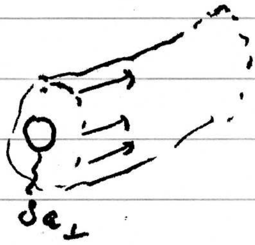
$$\underline{F} = q\underline{E} + q(\underline{v} \times \underline{B})$$



$$\underline{B} = \frac{\mu_0}{4\pi}\, \frac{q\, \underline{v} \times \hat{\underline{r}}_{12}}{r^2} \qquad \mu_0 = 4\pi \times 10^{-7}\ \frac{\text{henry}}{\text{m}}$$

CURRENT / AREA

$$\underline{J} = \frac{d\underline{I}}{da_\perp}$$



$$\underline{J} = \rho \underline{v} \qquad \rho = \frac{\text{moving charge}}{\text{volume}}$$

CONTINUITY EQUATION (Conservation of Charge)

$$\nabla \cdot \underline{J} = -\frac{\partial \rho}{\partial t}$$

GAUSS' LAW

$$\epsilon_0 \nabla \cdot \underline{E} = \rho$$

$$\underline{D} = \epsilon \epsilon_0 \underline{E}$$

UNIT CURRENT:

Ampere: $\dfrac{\underline{dl}_2 \times (\underline{dl}_1 \times \hat{r}_{12})}{r_{12}^2} = 1$

If $\delta f_{1\to 2} = 10^{-7}$ newtons

& $i_1 = i_2 = 1$ ampere

$\therefore\ \mu_0 = 4\pi \times 10^{-7}$

establishes "ampere"

AMPERES LAW WITH MAXWELL'S DISPLACEMENT CURRENT

BIOT – SAVART

$$\delta \underline{B} = \frac{\mu_0}{4\pi} \frac{\underline{i} \times \underline{\hat{r}}_{12}}{r_{12}^2} dl'$$

Differential form
{ Steady Currents
$\nabla \cdot \underline{J} = \frac{\partial \rho}{\partial t} = 0$

$$\underline{B}(\underline{r}) = \frac{\mu_0}{4\pi} \int_{(1)} \frac{\underline{J}(\underline{r}') \times \underline{\hat{r}}_{12}}{r_{12}^2} dv'$$

Integral form

DIRECTLY SHOW

$$\nabla \cdot \underline{B}(1) = 0$$

no assumptions!

AMPERES LAW

$$\nabla \times \underline{B}(\underline{1}) = \mu_0 \underline{J}(\underline{1})$$

Directly & using
$\nabla \cdot \underline{J} = 0$

see 5.3.2 of Griffiths (steady currents)
or 5.3 of Jackson

MAXWELL'S DISPLACEMENT CURRENT (1864–)

$$\nabla \cdot J = -\frac{\partial \rho}{\partial t}$$

$$\nabla \cdot \underline{J} = -\frac{\partial}{\partial t}(\epsilon_0 \nabla \cdot \underline{E}) = -\nabla \cdot \left(\epsilon_0 \frac{\partial \underline{E}}{\partial t}\right)$$

$$\nabla \cdot \left(\underline{J} + \epsilon_0 \frac{\partial \underline{E}}{\partial t}\right) \equiv 0$$

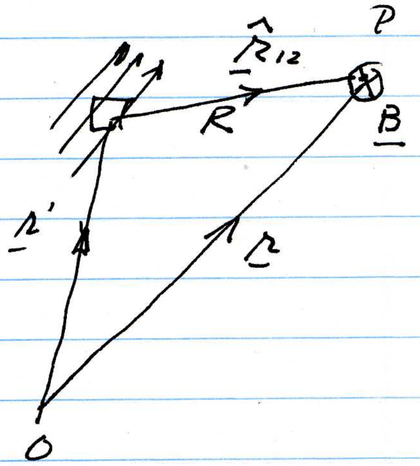
Propagation Hertz 1888

$$\boxed{\nabla \times \underline{H} = \underline{J} + \frac{\partial \underline{D}}{\partial t}}$$

$$\boxed{\nabla \times \underline{H} = \underline{J}_f + \frac{\partial \underline{D}}{\partial t}}$$

DIELECTRIC MATERIAL

Divergence & Curl of $\underline{B}$



$$\underline{B}(\underline{r}) = \frac{\mu_0}{4\pi} \int_{\text{allspace}} \frac{\underline{J} \times \hat{r}_{12}}{r^2}\, dv'$$

$\nabla\cdot$ operates @ $P(x,y,z)$

$$\nabla\cdot\underline{B} = \frac{\mu_0}{4\pi} \nabla_P\cdot\int = \frac{\mu_0}{4\pi}\int \nabla_P\cdot \frac{\underline{J}\times\hat{R}}{R^2}\, dv'$$

$$\nabla_P\cdot\left(\frac{\underline{J}\times\hat{R}}{R^2}\right) = \nabla_P\cdot\left(\underline{J}\times\frac{\hat{R}}{R^2}\right) \qquad \underline{J}(r') \neq f \; x,y,z$$

$$\nabla\cdot(\underline{a}\times\underline{b}) = \underline{b}\cdot(\nabla\times a) - a\cdot(\nabla\times b)$$

$$\nabla\cdot\frac{\underline{J}\times\hat{R}}{R^2} = \frac{\hat{R}}{R^2}\cdot\underbrace{(\nabla_P\times\underline{J})}_{=0} - \underline{J}\cdot\underbrace{\left(\nabla\times\frac{\hat{R}}{R^2}\right)}_{\text{just like } \nabla\times\frac{\hat{r}}{r^2} = 0}$$

$$\boxed{\nabla\cdot B(\underline{r}) \equiv 0}$$

$$\nabla\times\underline{B}(\underline{r}) = \frac{\mu_0}{4\pi}\int_{\text{allspace}} \nabla_P\times\left(\frac{\underline{J}\times\hat{r}_{12}}{r^2}\right) dv'$$

$$\nabla\times(a\times b) = \underbrace{\cancel{(b\cdot\nabla)a}}_{0} - (a\cdot\nabla)b + a(\nabla\cdot b) - \underbrace{\cancel{b(\nabla\cdot a)}}_{0}$$

$a = \underline{J}$, $b = \frac{\hat{r}_{12}}{R^2}$

$$= -(\underline{J}\cdot\nabla)\frac{\hat{R}}{R^2} + \underline{J}\left(\nabla\cdot\frac{\hat{R}}{R^2}\right)$$

$$= -\underline{J}\cdot\sum_x \hat{x}\frac{\partial}{\partial x}\frac{\hat{R}}{R^2} + \underline{J}\, 4\pi\,\delta(\underline{R})$$

$$-\sum J_x \frac{\partial}{\partial x}\left(\frac{\hat{R}}{R^2}\right) = 0$$

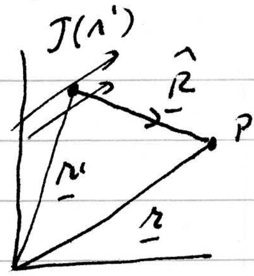
5.3.2 Griffiths

$\delta(x-\cancel{x'})\delta(y-\cancel{y'})\delta(z-\cancel{z'})\,d\cancel{x'}\,d\cancel{y'}\,d\cancel{z'}$

$\underline{J}(r') \to \underline{J}(\underline{r})$

$$\nabla\times\underline{B}(\underline{r}) = \frac{\mu_0}{4\pi}\int \underline{J}\, 4\pi\,\delta(\underline{R})\, dv'$$

$$\boxed{\nabla\times\underline{B}(\underline{r}) = \mu_0\,\underline{J}(\underline{r})}$$



JACKSON 5.3

$$\underline{B}(\underline{r}) = \frac{\mu_0}{4\pi} \int \frac{\underline{J}(\underline{r}') \times \hat{\underline{R}}}{R^2}\, dv'$$

$$\nabla_P \left(\frac{1}{R}\right) = -\frac{\hat{\underline{R}}}{R^2} = \frac{-\underline{R}}{R^3}$$

$$\frac{\mu_0}{4\pi} \underline{J}(\underline{r}') \times \frac{\hat{\underline{R}}}{R^2} = -\frac{\mu_0}{4\pi} \underline{J}(\underline{r}') \times \nabla\left(\frac{1}{R}\right)$$

$$\nabla \times (\psi \underline{a}) = \nabla\psi \times \underline{a} + \psi \nabla \times \underline{a} \underset{\underline{a}\ \text{const.}}{=} -\underline{a} \times \nabla\psi$$

$$\underline{B}(\underline{r}) = \frac{\mu_0}{4\pi} \int \nabla_P \times \frac{\underline{J}(\underline{r}')}{R}\, dv'$$

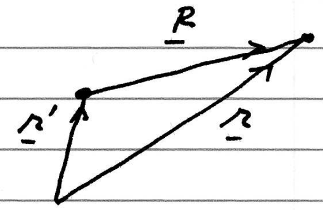
$$\underline{B}(\underline{r}) = \frac{\mu_0}{4\pi} \nabla_P \times \int \frac{\underline{J}(\underline{r}')}{R}\, dv'$$

Since $\underline{B}(\underline{r}) = \nabla \times$ Clearly $\underline{\nabla \cdot \underline{B} = 0}$

$$\nabla \times \underline{B}(\underline{r}) = \frac{\mu_0}{4\pi} \nabla \times \nabla \times \int \frac{\underline{J}(\underline{r}')}{R}\, dv'$$

VECTOR ANALYSIS IMPORTANT RESULTS

$$\nabla\cdot\left(\frac{\hat{\underline{R}}}{R^2}\right) = 4\pi\,\delta^3(\underline{R})$$



$$\nabla\cdot\left(\frac{\hat{r}}{r^2}\right) = 4\pi\,\delta^3(\underline{r})$$

$\underline{R} = \underline{r} - \underline{r}'$

∇ operates on $\underline{r}$

$$\nabla\times\left(\frac{\hat{\underline{R}}}{R^2}\right) = 0$$

$$\nabla\cdot(\nabla\times\underline{F}) = 0$$

$$\int \nabla\cdot\nabla\times\underline{F}\,dv = \int \nabla\times\underline{F}\cdot d\underline{a}$$
$$= \oint_{C_1} \underline{F}\cdot d\underline{l} + \int_{-C_1} \underline{F}\cdot d\underline{l} = 0$$

$$\nabla\times(\underline{A}\times\underline{B}) = (\underline{B}\cdot\nabla)\underline{A} - (\underline{A}\cdot\nabla)\underline{B} + A(\nabla\cdot B) - B(\nabla\cdot A)$$

$$\nabla\cdot(A\times B) = B\cdot(\nabla\times A) - A\cdot(\nabla\times B)$$

$$\nabla\times(\nabla\times A) = \nabla(\nabla\cdot A) - \nabla^2 A$$

$\nabla\cdot($

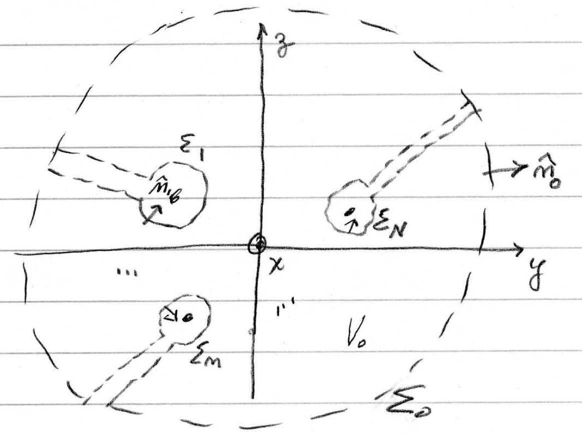
M
VA

ONE METHOD (TRADITIONAL) TO ISOLATE SINGULARITIES

PROVIDE SOURCE FREE VOLUME

DIVERGENCE THEOREM

$$\int \nabla \cdot \underline{F}\, dv = \int \underline{F} \cdot \underline{m}\, da$$



Volume V_0 WHICH CONTAINS SINGULARITIES, OUTER SURFACE Σ_0, INNER Σ_1 TO Σ_N

Enclosed singularities: surfaces $\Sigma_1, \cdots \Sigma_n, \cdots \Sigma_N$

Outward normals $\hat{m}_0, \hat{m}_1, \cdots \hat{m}_m, \cdots \hat{m}_N$

FUNCTIONS ϕ & ψ WITH NO SINGULARITIES IN $V_0 - \Delta V_1 \cdots - \Delta V_N$

FUNCTIONS AND FIRST & SECOND DERIVATIVES ARE CONTINUOUS IN

$$\int_{V_0 - \Delta V_1 \cdots - \Delta V_N} \nabla \cdot F\, dv = \sum_{\hat{m}_m = 0}^{N} \int \underline{F} \cdot \underline{\hat{m}}_m\, da$$

HERE WE HAVE GIVEN EXPLICIT MEANING TO

THE VOLUME INTEGRAL

AND THE SURFACE INTEGRAL

INSIDE $V_0 - \Delta V_1 - \Delta V_2 \cdots \Delta V_N$ Function $\underline{F}$ (F_x, F_y, F_z) Continuous + all derivates exist

SURFACES $\int \underline{F} \cdot m\, da$ NEEDS TO BE CALCULATED (ANY SIZE Σ_m IS OKAY NOT DIFFERENTIAL)

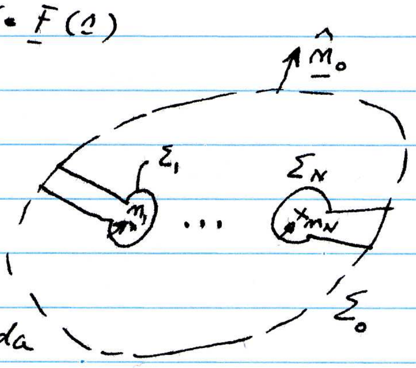
GAUSS' DIVERGENCE THEOREM WITH INTEGRABLE SINGULARITIES

CONSIDER OUTER SURFACE Σ_0, TOTAL INNER VOLUME V_0

INTEGRABLE SINGULARITIES IN $\nabla \cdot \underline{F}(\underline{r})$

Unit normals all outward directed

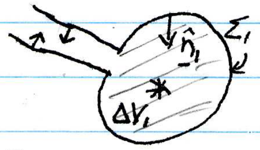
STANDARD FORM



$$\int_{V_0 - \Delta V_1 \cdots - \Delta V_N} \nabla \cdot \underline{F} \, dv = \sum_{m=0}^{N} \int \underline{F} \cdot \hat{\underline{m}}_m \, da$$

$$\int_{V_0 - \Delta V_1 \cdots - \Delta V_N} \nabla \cdot \underline{F} \, dv = \sum_{m=1}^{N} \int \underline{F} \cdot \hat{\underline{m}}_m \, da + \int_{\Sigma_0} \underline{F} \cdot \hat{n}_0 \, da$$

$$-\int_{\Sigma_1} \underline{F} \cdot \hat{m}_1 \, da = \int_{\Sigma_1} \underline{F} \cdot (-\hat{m}_1) \, da$$



$\parallel$

Use DIVERGENCE DEF Σ_1 TO

$$\int_{\Delta V_1} \nabla \cdot \underline{F}(\underline{r}) \, dv$$ NOTING δ-FUNCTION IS INTEGRABLE

$$\int_{V_0 - \Delta V_1 \cdots - \Delta V_N} \nabla \cdot \underline{F} \, dv + \int_{\Delta V_1 + \Delta V_2 \cdots + \Delta V_N} \nabla \cdot \underline{F} \, dv = \int_{\Sigma_0} \underline{F}(\underline{r}) \cdot \hat{\underline{m}}_0 \, da$$

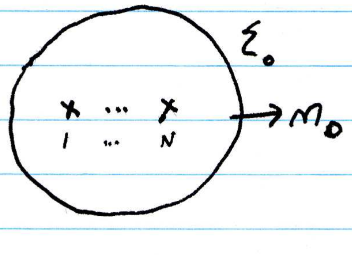
THIS GIVES US AN ALTERNATE FORM OF GAUSS' DIVERGENCE TH:

$$\int_{V_0} \nabla \cdot \underline{F}(\underline{r}) \, dv = \int_{\Sigma_0} \underline{F}(\underline{r}) \cdot \hat{\underline{m}}_0 \, da$$

V_0 entire vol including singularities $\qquad$ Σ_0 outer closed surface

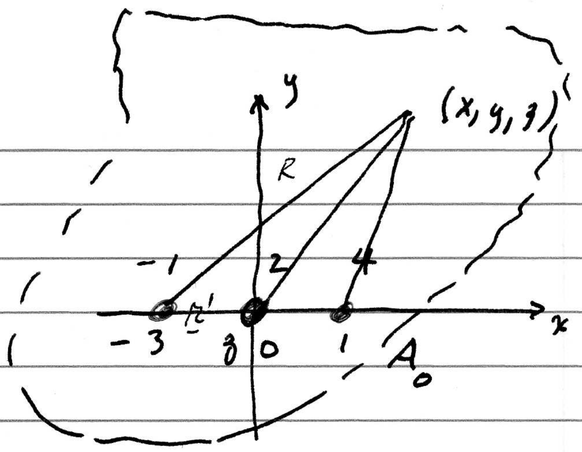
IT IS CONVENIENT TO USE WITH GREEN'S FUNCTION PROBLEMS

Singularities: $1 \ldots N$



Illustrative Problem

Consider charges shown



Calculate $\displaystyle\int_{A_0} \underline{E}\cdot\underline{n}\, da$

when A_0 is a shape shown enclosing all 3 points

Using extended Gauss' DT, we write $\qquad \underline{R} = \underline{r} - \underline{r}'$

$$\int_{A_0} \underline{E}\cdot\underline{n}\, da = \int_{V_0} \nabla\cdot\underline{E}\, dv = \int \frac{\rho}{\epsilon_0}\, dv \qquad dv = d^3x$$

$$\nabla\cdot\underline{E} = \frac{\rho(\underline{r})}{\epsilon_0} = \frac{\left[2\,\delta(\underline{r}-0) + 4\,\delta(\underline{r}-1\hat{x}) - 1\,\delta(\underline{r}+3\hat{x})\right]}{\epsilon_0}$$

$$\int_{A_0} \underline{E}\cdot\underline{n}\, da = \frac{1}{\epsilon_0}(+5)$$

1.5 Morse & Feshbach

Helmholtz's Theorems

Jan 03

26, 51 Weatherburn

Consider $\underline{F}$ vector point function, vanishing @ ∞

Helmholtz showed, $\underline{F}$ can be expressed $\begin{cases} \text{lamellar (express by potential)} \\ \text{solenoidal (express curl solenoid)} \end{cases}$

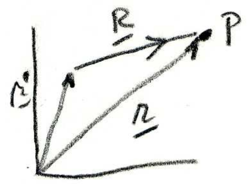
$$\underline{F} = \nabla\phi + \nabla\times\underline{H}$$

Helmholtz' Partition Theorem

"Solenoidal" vector means its $\nabla\cdot = 0$,
lamellar vector means its $\nabla\times = 0$

1) For ϕ: $\boxed{\nabla\cdot\underline{F} = \nabla^2\phi} + \underbrace{\cancel{\nabla\cdot\nabla\times\underline{H}}}_{0}$

$$\phi = -\frac{1}{4\pi}\int_{\substack{\text{all} \\ \text{space or where } \underline{F}\neq 0}} \frac{(\nabla\cdot\underline{F})'}{R}\,dv'$$



To ϕ can add any θ harmonic, i.e, $\nabla^2\phi = 0$

Since $\nabla^2(\phi+\theta) = \nabla^2\phi + \underbrace{\cancel{\nabla^2\theta}}_{0} = \nabla\cdot\underline{F}$ same

So ϕ not unique

2) For $\underline{H}$:

$$\nabla\times\underline{F} = \underbrace{\cancel{\nabla\times\nabla\phi}}_{0} + \nabla\times(\nabla\times\underline{H})$$

$$\nabla\times\underline{F} = \nabla(\nabla\cdot\underline{H}) - \nabla^2\underline{H}$$

Let $\underline{H}$ be solenoidal, i.e. $\nabla\cdot\underline{H} = 0$

$$\nabla\times\underline{F} = -\nabla^2\underline{H}$$

As before $\underline{H} = \frac{1}{4\pi}\int_{\text{all space}} \frac{\nabla\times\underline{F}}{R}\,dv'$

problem 16
prove $\nabla\cdot\underline{H} = 0$

$\boxed{\text{Restate: Given } \nabla\cdot\underline{F} \text{ \& } \nabla\times\underline{F} \text{ find } \phi, \underline{H} \text{ and find } \underline{F}}$

$$\underline{F} = -\frac{1}{4\pi}\nabla\int\frac{(\nabla\cdot\underline{F})'}{R}\,dv' + \frac{1}{4\pi}\nabla\times\int\frac{(\nabla\times\underline{F})'}{R}\,dv'$$

If $\nabla\cdot\underline{F} = 0$ (Solenoidal), then $\underline{F} = \nabla\times$ of some vector

If $\nabla\times\underline{F} = 0$ (Lamellar), then $\underline{F} =$ gradient of some scalar

HELMHOLTZ'S ... CONT

Jan 03

Sect 51 – Weatherburn

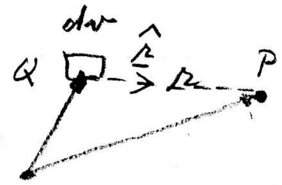
Consider Gn: $\nabla \cdot \underline{V}$, $\nabla \times \underline{V}$

Can be velocity of an incompressible fluid, i.e., $\nabla \cdot \underline{V} = 0$

From before

$$\underline{V} = -\frac{1}{4\pi}\cancel{\nabla \int \frac{\nabla \cdot \underline{V}}{r} dv} + \frac{1}{4\pi}\nabla \times \int \frac{\nabla \times \underline{V}}{r} dv = \nabla \times \underline{H}$$

$\underline{H}$ is A VECTOR POTENTIAL



Consider the $\underline{V}$ represented by $\underline{H}$

$$\underline{V} = \frac{1}{4\pi}\nabla_P \times \int \frac{\underline{W}}{r} dv$$

$\underline{W}$ is funct of coords of Q (not P)

$$\nabla_P \times \frac{\underline{W}}{r} = \nabla_P \frac{1}{r} \times \underline{W} + \frac{1}{r}\cancel{\nabla_P \times \underline{W}}^{\,0} = -\frac{\hat{r} \times \underline{W}}{r^2}$$

$$\underline{V} = \frac{1}{2\pi}\int \frac{\underline{W} \times \hat{r}}{r^2} dv$$

fluid 2× ?

$$\Delta \underline{V} = \frac{\mu \, d\underline{l} \times \hat{r}}{2\pi \; r^2}$$

{ like magnetic fld @ P due to differential current at Q

Griffith: App B: Longer Discussion of Unique or Not

$$\underline{F} = -\nabla U + \nabla \times \underline{W} = \frac{-1}{4\pi}\nabla \int \frac{\nabla \cdot \underline{F}}{r'} dv' + \frac{1}{4\pi}\nabla \times \int \frac{\nabla \times F}{r'} dv'$$

We can add to $\underline{F}$ any vector $\underline{G}$ with $\nabla \cdot \underline{G} = 0$ & $\nabla \times \underline{G} = 0$

With the requirement that $\underline{F} \to 0$ @ ∞; there is only $\underline{G} \equiv 0$ everywhere

With this requirement $\underline{F}$ is unique

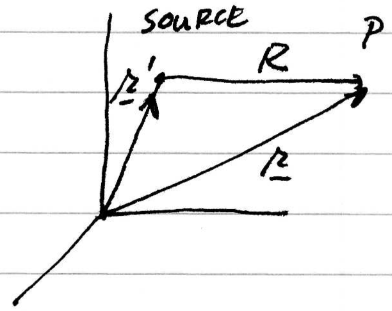
Otherwise $\underline{F}$ soln, also $\underline{F} + 8e_x$ also soln.

[For usual purposes: Helmholtz construction is unique

$8e_x$ $\quad \nabla \cdot 8\underline{e}_x = e_x \frac{\partial}{\partial x} \cdot e_x = 0$

$$\nabla \times \underline{e}_x = \sum e_x \frac{\partial}{\partial x} \times e_x = \underline{e}_z \frac{\partial 1}{\partial y} - \frac{\partial 1}{\partial y} e_z$$

$$\underline{F} = \frac{1}{4\pi}\nabla\int\frac{(\nabla\cdot\underline{F})'}{R}\,dv' + \frac{1}{4\pi}\nabla\times\int\frac{(\nabla\times\underline{F})'}{R}\,dv'$$



ILLUSTRATION

Gm: $\nabla\times\underline{F}$ & $\nabla\cdot\underline{F}$ Find $\underline{F}$

I. For a point charge at the origin

$$\underline{E},\ \phi \qquad \underline{E} = -\nabla\phi$$

$$\nabla\cdot\underline{E} = -\nabla^2\phi = \frac{\rho}{\epsilon_0}$$

We are given $\nabla\cdot\underline{E} = \delta(\underline{r})$

$$\nabla\times\underline{E} = 0$$

Find $\underline{E}$

$$\nabla\cdot\underline{E} = \delta(\underline{r}')$$

$$\underline{E} = \frac{1}{4\pi}\nabla\int\frac{\delta(\underline{r}')}{(\underline{r}-\underline{r}')}\,dv' + 0$$

$$\underline{E} = \frac{1}{4\pi}\nabla\left(\frac{1}{r}\right) = \frac{-1}{4\pi r^2}\hat{r}$$

UNITS

SYSTÈME INTERNATIONAL (SI) $\simeq$ MKS

Rationalized

$$\epsilon_0 = \frac{1}{36\pi \times 10^9} \ \frac{\text{farad}}{\text{meter}} \qquad c = \frac{1}{\sqrt{\mu\epsilon}} = 3 \times 10^8 \ \frac{m}{s}$$

$$\mu_0 = 4\pi \times 10^{-7} \ \frac{\text{henry}}{\text{meter}}$$

$$\sqrt{\frac{\mu_0}{\epsilon_0}} = 376.730 \ \Omega$$

Conductivity $\frac{1}{\text{ohm-meter}}$ $\qquad \underline{J} = \sigma \underline{E} \qquad \sigma \ \frac{\text{ampere}}{\text{volt meter}} = \frac{1}{\text{ohm-meter}}$

$S = \frac{1}{\text{ohm}} = \text{mho}$ 'Siemens' $\qquad$ Copper $1/1.68 \times 10^{-8} = 6. \times 10^7 \ S/m$

$$\underline{F} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r^2} \hat{r} \qquad \text{vs} \qquad \frac{q_1 q_2}{r^2} \hat{\underline{r}} \quad \text{Gaussian}$$

SI $\rightarrow$ Gaussian

set $\epsilon_0 = \frac{1}{4\pi}$

SOLVING MAXWELL'S EQUATIONS

IT IS IMPORTANT FOR THE STUDENT TO CATALOG THE MANY & VARIED APPROACHES TO SOLVING M.E. — BESIDES THE THEORY, THERE IS A GREAT DEAL TO LEARN BY CONSIDERING & RECONSIDERING THE MANY PROBLEMS THAT WE WILL SOLVE — IN CLASS AND IN PROBLEM SETS —

START WITH VECTOR FORMS OF M.E.

ME 1) $\nabla \times E(r,t)$
2) $\nabla \times H(r,t)$

$\xrightarrow{\mathcal{F}_t}$ / $\xrightarrow[HTD]{}$

MOST COMMON FORM

ME 1) $\nabla \times \underline{E}(r)$
2) $\nabla \times \underline{H}(r)$

$\downarrow$

CONVERT TO WAVE EQUATION IN $\underline{E}$ & $\nabla \times E$

$\downarrow$

SCALAR WAVE EQUATIONS

CONVERT TO $\left\{ \begin{array}{l} \text{WAVE EQUATION IN } \underline{E} \text{ \& } \nabla \times \underline{E} \\ \text{OR } \underline{H},\ \nabla \times \underline{H} \end{array} \right.$

THINK OVER HOW MANY USEFUL TECHNIQUES COME OUT OF THE USE OF FOURIER TRANSFORM THEORY

THINK OVER WHAT IT IS ABOUT VECTOR POTENTIAL FUNCTIONS THAT MAKES THEM USEFUL

$$\nabla^2 \mathcal{E}_y = \mu\varepsilon \frac{\partial^2 \mathcal{E}_y}{\partial t^2} \qquad\qquad \left(\nabla^2 + k^2\right) E_y(x,y,z;\nu) = 0$$

$$\begin{tikzcd}
\mathcal{E}_y(x,y,z,t) \arrow[r, "\mathcal{F}_t", "{\int - e^{-i2\pi\nu t}dt \ \text{(or htd)}}"'] \arrow[dr, "\mathcal{F}_{xyt}"'] & E_y(x,y,z;\nu) \arrow[d, "\mathcal{F}_{xy}"] \\
 & V_y\left(f_x, f_y; z; \nu\right)
\end{tikzcd}$$

$$\frac{\partial^2 V_y}{\partial z^2} + \left[k^2 - \left(2\pi f_x\right)^2 - \left(2\pi f_y\right)^2\right] V_y = 0$$

$$V_y\left(f_x, f_y; z; \nu\right) = \iint_{-\infty}^{\infty} dx\,dy\,E_y(x,y,z;\nu)\,e^{-i2\pi f_x x - i2\pi f_y y}$$

$$E_y(x,y,z;\nu) = \iint_{-\infty}^{\infty} df_x\,df_y\,V_y\left(f_x, f_y; z; \nu\right) e^{+i2\pi f_x x + i2\pi f_y y}$$

Signal Representations in Fourier Optics
Differential Equations

Problem Set IV
OPT 461
Nicholas George

TRANSFORMS - NOTATION -

USES LATER.

LAPLACE TRANSFORM $X(s)$ of the function $x(t)$

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

Corresponding Inversion

$$x(t)\,H(t) = \frac{1}{2\pi i} \int_{s=\sigma - i\infty}^{\sigma + i\infty} X(s)\, e^{st}\, dt$$

FOURIER TRANSFORM $G(\nu)$ of $g(t)$

$$G(\nu) = \int_{-\infty}^{\infty} g(t)\, e^{-i2\pi\nu t}\, dt$$

Corresponding Inversion

$$g(t) = \int_{-\infty}^{\infty} G(\nu)\, e^{+i2\pi\nu t}\, d\nu$$

Consistent with

$$G(f_x, f_y) = \iint_{-\infty}^{\infty} g(x,y)\, e^{-i2\pi f_x x - i2\pi f_y y}\, dx\, dy$$

$$g(x,y) = \iint G(f_x, f_y)\, e^{i2\pi f_x x + i2\pi f_y y}$$

i.e.

$$G(f_x, f_y, \nu) = \iiint g(x,y,t)\, e^{-i2\pi f_x x - i2\pi f_y y - i2\pi \cancel{f_t t}\, \nu t}\, df_x\, df_y\, dt$$

$$g(x,y,t) = \iiint G(f_x, f_y, \nu)\, e^{i2\pi f_x x + i2\pi f_y y + i2\pi\nu t}\, df_x\, df_y\, d\nu$$

MAXWELL'S EQUATION IN $\mathcal{FT}$ FORM

$$\mathcal{F}_t\left(\nabla \times \underline{\mathcal{E}}(\underline{r},t) = -\frac{\partial \mathcal{B}(r,t)}{\partial t} \right)$$

We aren the boundary values as we transform both sides w/og noting that the boundary conditions can be fit later

Letting $$\underline{E}(\underline{r};\nu) = \int_{-\infty}^{\infty} \underline{\mathcal{E}}(\underline{r},t)e^{-i2\pi\nu t}\,dt$$

Corresponding Inverse $$\underline{\mathcal{E}}(\underline{r},t) = \int_{-\infty}^{\infty} \underline{E}(\underline{r};\nu)e^{i2\pi\nu t}\,d\nu$$

$$\boxed{\begin{aligned} &\bullet\quad \nabla \times \underline{E}(\underline{r};\nu) = -i2\pi\nu\,\underline{B}(\underline{r},\nu) \\ &\text{Similarly } \bullet\quad \nabla \times \underline{H}(\underline{r};\nu) = \underline{J}(\underline{r};\nu) + i2\pi\nu\,\underline{D}(\underline{r};\nu) \end{aligned}}$$

can be derived from $$\mathcal{F}_t\left(\nabla \times \mathcal{H}(\underline{r},t) = \underline{\mathcal{J}}(\underline{r},t) + \frac{\partial \mathcal{D}(\underline{r},t)}{\partial t} \right)$$

TIME VARYING EM FIELDS

WAVE EQUATION

Balanis 3.1 to 3.3

$$\nabla \times \underline{\mathcal{E}}(\underline{r},t) = -\frac{\partial \mathcal{B}(\underline{r},t)}{\partial t}$$

$$\nabla \times \underline{H}(\underline{r},t) = \underline{J} + \frac{\partial \mathcal{D}}{\partial t}$$

Consider ~~source~~ charge free region: $\rho = 0$

Homogeneous

no magnetic poles

$$\left.\begin{aligned} \mathcal{B} &= \mu H \\ \mathcal{D} &= \epsilon \underline{\mathcal{E}} \\ \mathcal{J} &= \sigma \underline{\mathcal{E}} \quad \text{ohmic} \end{aligned}\right\}$$

define a simple medium for EM topics

$$\nabla \times \underline{\mathcal{E}} = -\mu \frac{\partial H}{\partial t}$$

$$\nabla \times \underline{H} = \sigma \mathcal{E} + \epsilon \frac{\partial \mathcal{E}}{\partial t} = \mathcal{J} + \epsilon \frac{\partial \mathcal{E}}{\partial t}$$

$$\nabla \times \nabla \times \underline{\mathcal{E}} = -\mu \frac{\partial}{\partial t} \nabla \times \underline{H} = -\mu\sigma \frac{\partial \mathcal{E}}{\partial t} - \mu\epsilon \frac{\partial^2 \mathcal{E}}{\partial t^2}$$

$$= -\mu \frac{\partial \mathcal{J}}{\partial t}$$

$$\nabla \times \nabla \times \underline{\mathcal{E}} + \mu\sigma \frac{\partial \mathcal{E}}{\partial t} + \mu\epsilon \frac{\partial^2 \mathcal{E}}{\partial t^2} = 0$$

$$\underbrace{\nabla(\nabla \cdot \underline{\mathcal{E}})}_{=0} - \nabla^2 \underline{\mathcal{E}} + \mu\sigma \frac{\partial \mathcal{E}}{\partial t} + \mu\epsilon \frac{\partial^2 \mathcal{E}}{\partial t^2} = 0$$

$$\nabla^2 \underline{\mathcal{E}} = \mu\sigma \frac{\partial \mathcal{E}}{\partial t} + \mu\epsilon \frac{\partial^2 \mathcal{E}}{\partial t^2} \quad ✓ \quad (3.12)$$

or

$$\nabla \times \nabla \times \underline{\mathcal{E}} + \mu\epsilon \frac{\partial^2 \mathcal{E}}{\partial t^2} = -\mu \frac{\partial \mathcal{J}}{\partial t} \quad ✓$$

EM THEORY : WAVE EQUATION

Jan 19, 2005
N. George

Starting with Maxwell's Equations –

1) $$\nabla \times \underline{\mathcal{E}}(\underline{r},t) = -\frac{\partial \underline{B}(\underline{r},t)}{\partial t}$$

2) $$\nabla \times \underline{H}(\underline{r},t) = \underline{J}(\underline{r},t) + \frac{\partial \underline{D}(\underline{r},t)}{\partial t}$$

Our objective today is to derive the basic wave equation for $\underline{\mathcal{E}}(\underline{r},t)$ combining 1) + 2). Then "a solution of Maxwell's Eqs" is found by solving

$$\nabla \times \nabla \times \underline{\mathcal{E}}(\underline{r},t) + \mu\epsilon \frac{\partial^2 \underline{\mathcal{E}}(\underline{r},t)}{\partial t^2} = -\mu \underline{J}(\underline{r},t)$$

and then using (1) to find ... $\underline{B}(\underline{r},t)$

As an alternate way of proceeding one can use 1) + 2) to find wave equation for $\underline{H}(\underline{r},t)$. Solve this for $\underline{H}(\underline{r},t)$ & plug in to (2) ... for $\underline{D}(\underline{r},t)$

Another objective today: Boundary conditions for $\underline{\mathcal{E}}, \underline{H} \,\big|_{\text{tangential}}$ 1, 2

And the divergence conditions for $\underline{B}, \underline{D} \,\big|_{\text{normal}}$ (3), (4)

$$\nabla^2 \underline{\mathcal{E}}(\underline{r},t) - \mu\epsilon \frac{\partial^2 \underline{\mathcal{E}}(\underline{r},t)}{\partial t^2} = \mu \frac{\partial \underline{J}(\underline{r},t)}{\partial t} + \nabla\left(\frac{\rho}{\epsilon}\right)$$

OHMIC MEDIUM

$$\nabla^2 \underline{\mathcal{E}}(\underline{r},t) - \mu\epsilon \frac{\partial^2 \underline{\mathcal{E}}(\underline{r},t)}{\partial t^2} = \mu\sigma \frac{\partial \underline{\mathcal{E}}}{\partial t} + \nabla\left(\frac{\rho}{\epsilon}\right)$$

WAVE EQUATION

Solution Pair to solve Maxwell's Equations

$$\nabla^2 \underline{\mathcal{E}} - \mu\epsilon \frac{\partial^2 \underline{\mathcal{E}}}{\partial t^2} = \mu \frac{\partial \underline{J}}{\partial t} + \nabla\left(\frac{\rho}{\epsilon}\right)$$

&

$$\nabla \times \underline{\mathcal{E}} = -\mu \frac{\partial \underline{H}}{\partial t}$$

Similarly, with 3-13

$$\nabla^2 \underline{H} = \mu\sigma \frac{\partial \underline{H}}{\partial t} + \mu\epsilon \frac{\partial^2 \underline{H}}{\partial t^2}$$

we would pair

$$\nabla \times \underline{H} = \sigma\mathcal{E} + \frac{\epsilon\,\partial \mathcal{E}}{\partial t}$$

Continuity -

$$\nabla \cdot \underline{J} + \frac{\partial \rho}{\partial t} = 0$$

18

24 Jan 2005

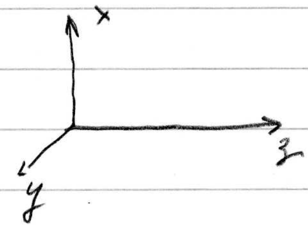
Review last lecture: Boundary Conditions
Wave Equation

Illustrative Problem

$$\nabla^2 \underline{\mathcal{E}} = \mu\sigma \frac{\partial \underline{\mathcal{E}}}{\partial t} + \mu\epsilon \frac{\partial^2 \underline{\mathcal{E}}}{\partial t^2}$$

LETS FIND A SIMPLE SOLUTION TO ESTABLISH THAT EM/Light propagates

Consider a $\hat{z}$-directed polarized wave $\hat{x}$



Test solution $\underline{\mathcal{E}} = \hat{x}\, g(z - vt)$

$$\nabla^2 \mathcal{E}_x = \mu\sigma \frac{\partial \mathcal{E}_x}{\partial t} + \mu\epsilon \frac{\partial^2 \mathcal{E}_x}{\partial t^2}$$

$$\frac{\partial \mathcal{E}_x}{\partial z} = g'(z-vt) \qquad \frac{\partial \mathcal{E}}{\partial t} = -v\, g'(z-vt)$$

$$\frac{\partial^2 \mathcal{E}_x}{\partial z^2} = g''(z-vt) \qquad \frac{\partial^2}{\partial t^2} = v^2 g''(z-vt)$$

$$g''(z-vt) = \mu\sigma(-v)g'(z-vt) + \mu\epsilon v^2 g''$$

Take $\sigma = 0$: We find $\quad g'' = \mu\epsilon v^2 g''$

$$v^2 = \frac{1}{\mu\epsilon}$$

$$\boxed{v = \pm \frac{1}{\sqrt{\mu\epsilon}}}$$

WAVE EQUATIONS / SOLUTIONS

Kong 1.2

FREE SPACE – SOURCE FREE

(1) $\nabla \times \bar{E} = -\mu \dfrac{\partial \bar{H}}{\partial t}$ $\qquad \nabla \cdot \bar{H} = 0$

(2) $\nabla \times \bar{H} = -\epsilon \dfrac{\partial \bar{E}}{\partial t}$ $\qquad \nabla \cdot \bar{E} = 0$

(3) $\nabla^2 \bar{E} - \mu\epsilon \dfrac{\partial^2 \bar{E}}{\partial t^2} = 0$

SIMPLEST SOLUTION: $\bar{E} = \hat{x}\, E_0 \cos(kz - \omega t) = \hat{x}\, E_x(z,t)$

Substitute into WE see

$$k^2 = \omega^2 \mu\epsilon$$

See propagation $k\Delta z - \omega \Delta t = 0$

$$v = \frac{\Delta z}{\Delta t} = \frac{\omega}{k} = \frac{\omega}{\omega\sqrt{\mu\epsilon}} = \frac{1}{\sqrt{\mu\epsilon}}$$

Wavelength $k = \dfrac{2\pi}{\lambda}$

FIND $\underline{H}$ by (1): $\bar{H} = \hat{y}\sqrt{\dfrac{\epsilon}{\mu}}\, E_0 \cos(kz - \omega t)$

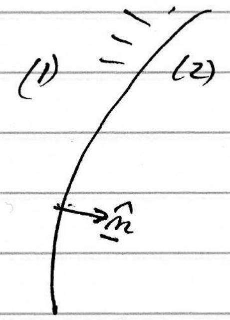
$\left|\dfrac{E}{H}\right| = \sqrt{\dfrac{\mu}{\epsilon}} = \eta$ characteristic impedance $377\,\Omega$

$\bar{E} \perp \bar{H} \perp \bar{v}$

WAVENUMBER: $k = \dfrac{2\pi}{\lambda} = \dfrac{2\pi}{c}\nu$ $\qquad$ Fundamental importance

Boundary Conditions

Summary: From M.E. for $\nabla \times \underline{E}(\underline{r},t)$ & $\nabla \times \underline{H}(\underline{r},t)$



$$\left.\begin{aligned} E_{1\,\text{tangential}} &= E_{2\,\text{tangential}} \\ H_{1\,\text{tangential}} &= H_{2\,\text{tangential}} \end{aligned}\right\}$$

equivalent expressions are

$$\left.\begin{aligned} \hat{n} \times (\underline{E}_2 - \underline{E}_1) &= 0 \\ \hat{\underline{n}} \times (H_2 - H_1) &= 0 \end{aligned}\right\} \quad \text{Balanis } 1\text{-}26a, \ 1\text{-}27a$$

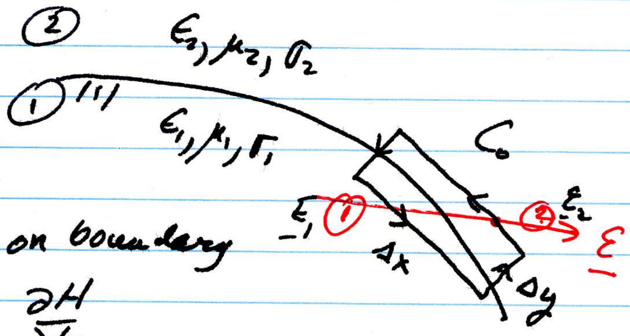
Dependent Conditions

D_n continuous simple μ, ϵ, σ
B_n

BOUNDARY CONDITIONS

Balanis 1.5.1

DESCRIBE INTERFACE



No placed charges on boundary

$$\nabla \times \mathcal{E} = -\mu \frac{\partial H}{\partial t}$$

Closed curve C_o, area S_o $\Delta y \to 0$

$$\int_{S_o} \nabla \times \underline{\mathcal{E}} \cdot \underline{n}'_{loop} \, da = -\int_{S_o} \mu \frac{\partial \underline{H}}{\partial t} \cdot n' da$$

$n' = \perp$ loop. in surface

|| Stoke's Th

$$\int_{C_o} \underline{\mathcal{E}} \cdot d\underline{l} = \mu \int_{S_o} \frac{\partial \underline{H}}{\partial t} \cdot \underline{n} \, da$$

$$\mathcal{E}_1 \cdot \hat{x} \, \Delta x - \mathcal{E}_2 \cdot \hat{x} \, \Delta x + (\cdot) \Delta y = -\mu \left. \frac{\partial H}{\partial t} \right|_n \Delta x \Delta y$$

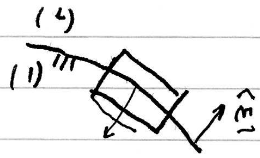
$$\underline{\mathcal{E}}_1 \cdot \underline{\hat{x}} = \underline{\mathcal{E}}_2 \cdot \underline{\hat{x}}$$

$$\mathcal{E}_{1\,tangential} = \mathcal{E}_{2\,tangential}$$

define $\hat{n}$ as unit normal to ~~area loop~~ surface

$$\hat{n} \times (\underline{\mathcal{E}}_2 - \underline{\mathcal{E}}_1) = 0 \qquad 1\text{-}26a$$

Boundary Conditions



From $\quad \nabla \times \underline{H} = \underline{J} + \dfrac{\partial \mathcal{D}}{\partial t}$

$$\int \nabla \times \underline{H} \cdot \underline{m}\, da = \int \underline{J} \cdot \underline{m}\, da + \int \epsilon \frac{\partial \mathcal{E}}{\partial t} \cdot \underline{m}\, da$$

analogously, we see that (finite ϵ, μ, σ)

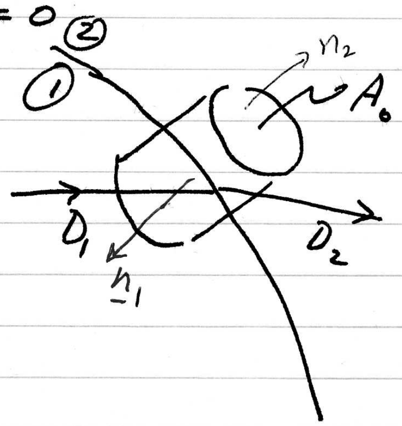
$$\hat{\underline{m}} \times (H_2 - H_1) = 0$$

or

$$H_{1\,tangential} = H_{2\,tangential} \qquad 1\text{-}27a$$

From Divergence Conditions

$$\nabla \cdot \underline{D} = \rho_{placed} = 0$$



$\int \nabla \cdot \underline{D}\, dv = 0$

$\parallel$ Gauss 1777-1855

$\int \underline{D} \cdot \underline{m}\, da$

$\parallel$

$$\underline{D}_1 \cdot \underline{m}_1 - \underline{D}_2 \cdot \underline{m}_1 = 0$$

$$\boxed{D_{1\,normal} = D_{2\,normal}} \qquad \begin{cases} 1\text{-}29a \\ 1\text{-}30a \end{cases}$$

Analogously: $\nabla \cdot \underline{B} = 0 \Rightarrow$

$$B_{1\,normal} = B_{2\,normal} \qquad \begin{cases} 1\text{-}31a \\ 1\text{-}32a \end{cases}$$

Conductors

Consider large (unbounded) medium μ, σ, ϵ in which σ is large. Imagine at $t=0$, we establish near the origin some placed charge. Provide analytical explanation

Assert Ohm's Law

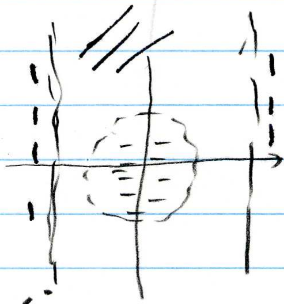
$$\underline{J} = \sigma \underline{E}$$

1) $\nabla \times \underline{E} = -\frac{\partial B}{\partial t}$

2) $\nabla \times \underline{H} = J + \frac{\partial D}{\partial t}$

$$0 = \nabla \cdot \nabla \times \underline{H} = \nabla \cdot \underline{J} + \epsilon \frac{\partial \nabla \cdot E}{\partial t}$$

Do we want history of $\underline{J}$ or ρ?



First notice that 1), 2) yield continuity equation:

$$\nabla \cdot \underline{J} + \frac{\partial \rho}{\partial t} = 0$$

which is okay since we know this was used to derive ME in the first place

$$\boxed{\frac{\partial \rho}{\partial t} + \left(\sigma \nabla \cdot E = \frac{\sigma \rho}{\epsilon}\right) = 0}$$

$$\rho(\underline{r}, t) = \rho(\underline{r}, 0) e^{-\frac{\sigma t}{\epsilon}}$$

Ref: Table 2.3 & p61 Balanis

The decay time for charge is $t_r = \frac{\epsilon}{\sigma} =$

Copper

$$t_r = \frac{\epsilon}{\sigma} = \frac{8.854 \times 10^{-12}}{5.76 \times 10^{7}}$$

$$t_r = 1.54 \times 10^{-19}\,\mathrm{s}$$

vs

Glass

$$t_r = 6\left(\frac{8.854 \times 10^{-12}}{10^{-12}}\right)$$

$$t_r = 53\,\mathrm{sec}$$

CONDUCTORS -

Jan 2005

For the present, we are not going into the details of the constitutive parameters : ϵ, μ, σ Ch 2 of Balanis
If you did not study this material in your introductory course, you need to review it in the next few weeks.
We take ϵ, μ, σ to be constant scalars - Simple Media
We would like to study conductors & ME by the following:

For understanding plane wave propagation in lossy media } consider
boundary condition at perfect conductor } example.

i) Suppose we place a charge inside of a perfect conductor, What happens

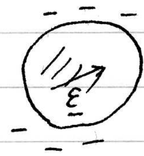
$$\nabla \times \underline{\mathcal{E}} = -\frac{\partial B}{\partial t}$$

$$\nabla \cdot \underline{\mathcal{E}} = \rho/\epsilon_0$$

ohmic medium

$$\underline{J} = \sigma \underline{E}$$

Recall: Electrostatics
We charge a metallic Sphere by $-Q$
& solve Laplace's Equation



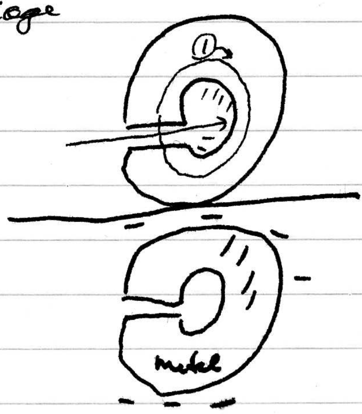
as $\sigma \to \infty$
the observed absence of infinite $\underline{J}$ tells us that $\underline{\mathcal{E}}_{interior} = 0$
Using Gauss' law, we see that $\underline{E}_{interior} = 0$

in steady state Put --- inside a "Faraday" cage

$$\int_{①} \underline{E} \cdot d\underline{s} = \begin{cases} Q_- \\ 0 \end{cases}$$

Even though we place Q_- inside, it must migrate to outer surface

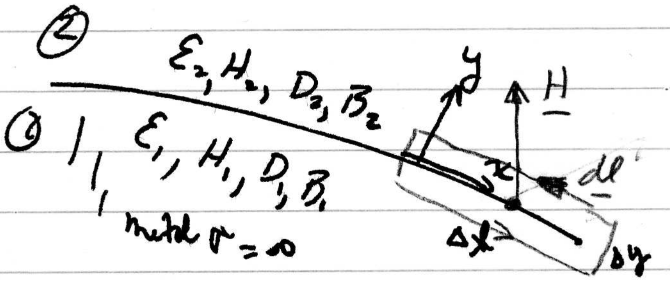
quantitatively inside $\rho = 0$
$\underline{\mathcal{E}} = 0$



Boundary Condition - Perfect Conductor

$$\nabla \times \underline{\mathcal{E}} = -\frac{\partial \underline{B}}{\partial t}$$

$$\nabla \times \underline{H} = \underline{J} + \frac{\partial \underline{D}}{\partial t}$$



Consider medium 1 is a perfect conductor $\sigma_1 \to \infty$:

At least in some short time after start :

$$\mathcal{E}_1 = 0 \Rightarrow B_1 = 0$$

$$\int \nabla \times \underline{H} \cdot \underline{da} = \int \left(\underline{J} + \frac{\partial \underline{D}}{\partial t}\right) \cdot \underline{da}$$

$$\parallel$$

$$\int \underline{H} \cdot \underline{dl}$$

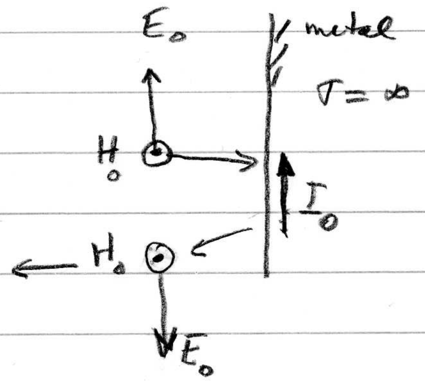
$$\parallel$$

delta function

$$H_{t_2} \cancel{\Delta l} = (J \delta y) \cancel{\Delta l} + \frac{\partial \underline{D}}{\partial t} \cancel{\Delta l} \, \delta y \qquad \underline{J} \delta y = I_1$$

$$H_{t_2} = I_0 \quad \text{use right hand rule}$$

In application, here is the story



For the incident E_0 & H_0,

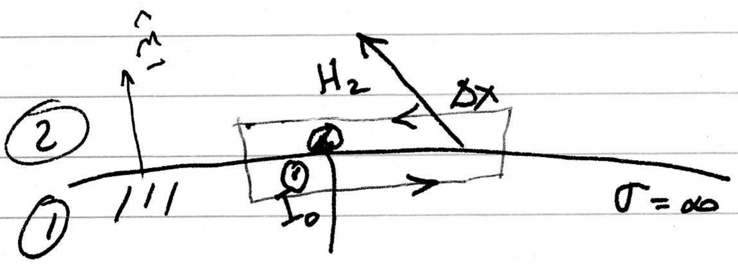
$$I_0 = 2 H_0$$

With the reflected wave

E_0, H_0 as shown

BOUNDARY CONDITION

PERFECT CONDUCTOR



$$\nabla \times \underline{H} = \underline{J} + \frac{\partial \underline{D}}{\partial t}$$

$$\int \underline{H} \cdot d\underline{l} = H_{2t}\,\Delta x = \underbrace{J \cdot \Delta x \Delta y}_{} + \frac{\partial D}{\partial t}\Delta x \Delta y$$

$$H_{2t}\,\Delta x = I_0\,\Delta x \qquad J_s \text{ Balanis uses}$$

$$H_{2t} = I_0 = J\Delta x \qquad 1.42$$

surface current

In an earlier lecture we ~~[illegible]~~ proved that in a perfect conductor

$$\underline{E} = 0$$

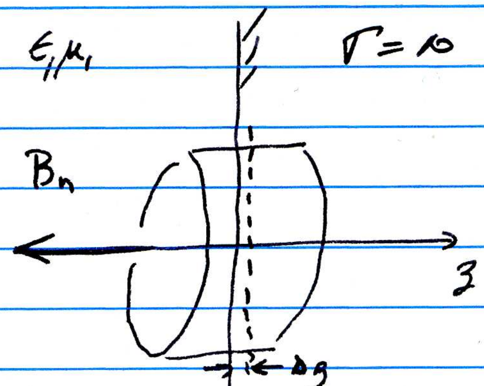
$$\underline{H} = 0$$

Perfect Conductor

E_{outside} must be normal

$\underline{E}$ inside $= 0$

$\nabla \cdot \underline{E} = \rho_{\text{TOTAL}}$ chge/vol



$\nabla \cdot \underline{D} = \rho_{\text{free}}$ { THIS INCLUDES CHARGES ON CONDUCTOR SURFACE (ASSERTION) Differently handled from polarization charges on dielectric!

CONSIDER THE PILL-BOX

$$\int \nabla \cdot \underline{D} \, dv = \int \rho \, dx \, dy \, dz$$

$$\parallel$$

$$\int \underline{D} \cdot d\underline{a} =$$

Let $\boxed{\rho(x,y,z) = Q_s \, \delta(z)}$

$Q_s = \dfrac{\text{charge}}{\text{area}}$

see $\Delta z \to 0$

$$D_n \, \Delta x \Delta y = Q_s \, \Delta y \Delta x = \int Q_s \, \delta(z) \, dx \, dy \, dz$$

$$\boxed{\begin{aligned} D_n &= Q_s \\ \epsilon E_n &= Q_s \end{aligned}}$$

From $\rho(x,y,z) = Q_s \, \delta(z)$

WATCH OUT FOR BALANIS UNCONVENTIONAL NOTATION

TABLE 1.5

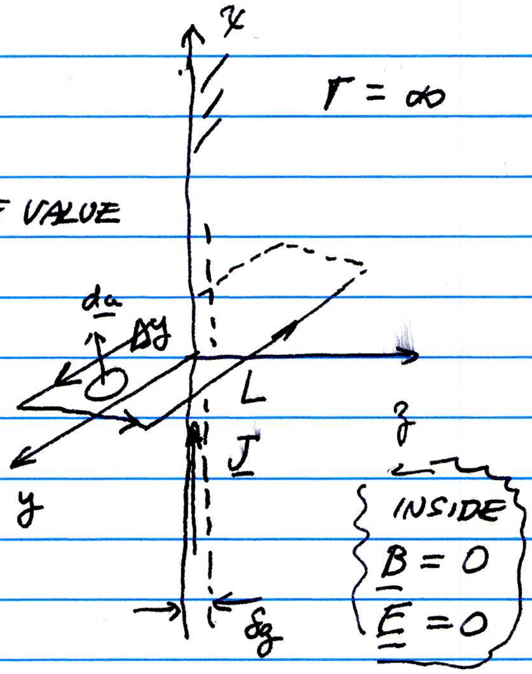
q_{es} TOO MUCH LIKE

$\rho_{\text{free}} = q_{ev}$ IN TABLE 1.4 } NOT SAME

ρ_{ev}

PERFECT CONDUCTOR - INTERFACE

$$\nabla \times \underline{H} = \underline{J} + \frac{\partial \underline{D}}{\partial t}$$



AS $\sigma \to \infty$ THE $J \to$ LARGE VALUE

CONCENTRATED AT SURFACE

CONSIDER LOOP SHOWN

H_t = tangen fiel

$$\int \nabla \times H \cdot da = \int \underline{J} \cdot d\underline{a}$$

$$\int_{LOOP} \underline{H} \cdot dl = \int J \, \Delta z \, \Delta y + \int \frac{\partial \underline{D}}{\partial t} \cdot d\underline{a}$$

LET $\underline{J}(x,y,z) = \hat{x} \, I_\ell \, \delta(z)$ $\qquad I_\ell = \frac{\text{current}}{\text{length}}$

$$H_t L = \iint I_\ell \, \delta(z) \, dz \, dy$$

$$H_t L = \int I_\ell \, dy = I_\ell L$$

$$\boxed{H_t = I_\ell}$$

units of

$$\underline{J}(x,y,z) = I_\ell \, \delta(z)$$

$$\frac{1}{\Delta z} \text{rect} \frac{z'}{\Delta z} = \frac{1}{\text{length}}$$

$$J \, da \qquad \frac{\text{current}}{\text{area}} \cdot \text{area} = \frac{\text{current}}{\text{length}} \frac{1}{\text{length}} \cdot \text{length} \cdot \text{length}$$

Important idea is in matching BC, if you use E_{tang}, H_{tang} that is it. The conditions on B_n & D_n are 'dependent' i.e. they follow from E_{tang}, H_{tang} and Maxwell's equations.

LECTURE 1 Feb 2006

TIME DEPENDENT

POYNTING'S THEOREM power out (t)

Illustrate sinusoidal case

average power out

REVIEW STORED ENERGY

WAVE EQUATION E, H SIMPLE CASE TO ILLUSTRATE

$$v = \pm \frac{1}{\sqrt{\mu \epsilon}} \quad \&$$

Close out Ch 1 noting all x, y, z, t or $\underline{r}, t$ dependence
Signal Representation is next –

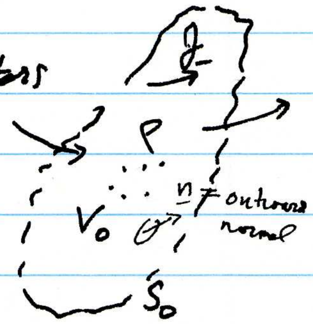
$\left.\begin{array}{l}\text{Fourier Xfm} \\ \text{Laplace Xfm} \\ e^{j\omega t} \text{ SS}\end{array}\right\}$ methods of solving

Energy Equations - Poynting's Theorem - Time Dependent Form

Let us reflect a moment as to where to start in considering an energy equation — i) We know from circuit theory that $V \cdot I$ is a good start
ii) We know that we must use Maxwell's equations too

(1) $$\nabla \times \underline{\mathcal{E}} = -\frac{\partial \underline{B}}{\partial t}$$

all vectors



(2) $$\nabla \times \underline{\mathcal{H}} = \underline{J} + \frac{\partial \underline{D}}{\partial t}$$

Analogue to VI is $\underline{\mathcal{E}} \cdot \underline{J}$ $\qquad \frac{V}{m}\,\frac{a}{m^2} = \frac{\text{watts}}{\text{vol}}$

$$\underline{\mathcal{E}} \cdot \underline{J} = \underline{\mathcal{E}} \cdot \left(\nabla \times \underline{H} - \frac{\partial \underline{D}}{\partial t} \right) \quad \leftarrow \text{Using (1)}$$

Need to find $\nabla \times \underline{\mathcal{E}}$ somehow . . .

$$\nabla \cdot (\underline{\mathcal{E}} \times \underline{H}) = \underline{H} \cdot (\nabla \times \mathcal{E}) - \underline{\mathcal{E}} \cdot (\nabla \times \underline{H})$$

$$\underline{\mathcal{E}} \cdot \underline{J} = \underline{H} \cdot (\nabla \times \underline{\mathcal{E}}) - \nabla \cdot (\underline{\mathcal{E}} \times \underline{H}) - \underline{\mathcal{E}} \cdot \frac{\partial \underline{D}}{\partial t}$$

$$\rightarrow \nabla \cdot (\underline{\mathcal{E}} \times \underline{H}) + \underline{\mathcal{E}} \cdot \underline{J} + \underline{\mathcal{E}} \cdot \frac{\partial \underline{D}}{\partial t} + \mathcal{H} \cdot \frac{\partial \underline{B}}{\partial t} = 0 \quad \leftarrow \text{Using (2)}$$

Gauss' Th:

$$\int_{S_0} \underline{\mathcal{E}} \times \underline{H} \cdot \underline{n}\, da + \int_{V_0} \underline{\mathcal{E}} \cdot \underline{J}\, dv + \int \left[\underline{\mathcal{E}} \cdot \frac{\partial \underline{D}}{\partial t} + \underline{\mathcal{H}} \cdot \frac{\partial \underline{B}}{\partial t} \right] dv = 0$$

interpret this term

work by flow currents, incl joule heating

watts

time rate of increase in stored energy: electric + magnetic

$\frac{\text{joules}}{\text{sec}} = \text{watts}$

Asserting conservation of energy gives us the interpretation that the Poynting vector $\underline{S} = \underline{\mathcal{E}} \times \underline{H}$ when integrated over a closed surface, as above, gives us the net outflow of power at any t.

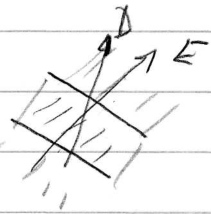
$$\frac{\text{instantaneous energy out of } S_0}{\text{time}} = \int_{S_0} \underline{\mathcal{E}} \times \underline{H} \cdot \underline{n}\, da$$

11.02	Sm
p1415	Papas
6.7	Jackson
1.6	Balanis
8.1.2	Griffiths

ENERGY STORED IN EM FIELD

Jan 2005

What is energy stored in electric field ?

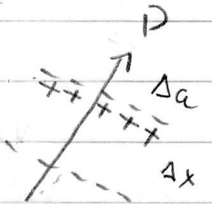


Consider differential vol :

$C = Q/V$

$U = \frac{1}{2} C V^2 = \frac{1}{2} Q V$ $\quad V = E \cdot \Delta x$

$\nabla \cdot \underline{D} = \rho$: charge/area $= D$



$U = \frac{1}{2} \frac{\underline{D} \cdot \underline{E} \, \Delta x \, \Delta a}{\Delta a \quad \Delta x}$ $\quad \frac{\text{energy}}{\text{vol}}$

$\frac{U}{\text{vol}} = \frac{1}{2} \underline{D} \cdot \underline{E}$

$\frac{\text{Energy density elec fld}}{\text{vol}} = \frac{1}{2} \underline{D} \cdot \underline{E}$

$\frac{1}{2} \epsilon \underline{E} \cdot E = \frac{1}{2} \epsilon |E|^2$

$\frac{\partial}{\partial t} = \frac{1}{2} \epsilon \, 2 |\underline{E}|$

$$\boxed{\frac{\partial}{\partial t} w_e = \underline{E} \cdot \frac{\partial D}{\partial t}}$$

2.07 SMYTHE

What is energy density of a magnetic field

332-3 Sm

$u_h = \frac{1}{2\mu} \underline{B} \cdot \underline{B} = \frac{1}{2} \underline{B} \cdot \underline{H}$

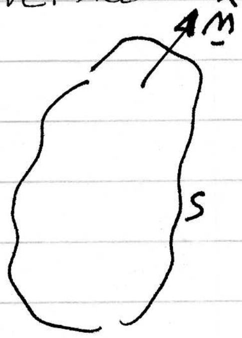
$$\boxed{\frac{\partial}{\partial t} w_m = \underline{H} \cdot \frac{\partial B}{\partial t}}$$

8.02 Sm.

energy stored in electric field per unit volume : $w_e = \frac{1}{2} \underline{D} \cdot \underline{E}$ $\quad$ at any instant t

energy density in magnetic field : $w_m = \frac{1}{2} \underline{B} \cdot \underline{H}$ $\quad$ p14/15 CAP

POYNTING VECTOR $\quad \mathcal{E}, \mathcal{H}$ notation $\quad$ TIME AVERAGE



Theorem states

Instantaneous: Power out flow $= \int_S \underline{\mathcal{E}} \times \underline{\mathcal{H}} \cdot \underline{n}\, da$

Suppose $\quad \underline{\mathcal{E}}(\underline{r},t) = \underline{\mathcal{E}}(\underline{r}) \cos \omega t \rightarrow |A| e^{i\omega t}$

$\underline{\mathcal{H}}(\underline{r},t) = \underline{\mathcal{H}}(\underline{r}) \cos(\omega t + \varphi) \rightarrow |B| e^{i(\omega t+\varphi)}$

$$\underline{\mathcal{E}} \times \underline{\mathcal{H}} = \underline{\mathcal{E}}(\underline{r}) \times \underline{\mathcal{H}}(\underline{r}) \cos(\omega t) \cos(\omega t + \varphi)$$

$$= \underset{A}{\underline{\mathcal{E}}(\underline{r})} \times \underset{B}{\underline{\mathcal{H}}(\underline{r})} \left[\tfrac{1}{2}\right]\left[\cos\varphi + \cos(2\omega t + \varphi)\right]$$

TIME AVERAGE

$$\overline{\underline{\mathcal{E}} \times \underline{H}} = \underset{A}{\underline{\mathcal{E}}} \times \underset{B}{\underline{\mathcal{H}}} \; \frac{\cos\varphi}{2}$$

$A \quad i(\omega t) \quad -i(\omega t+\varphi) \quad -i\varphi$

OPTICS 462: Electromagnetic Theory and Fourier Optics

OVERVIEW

Let us discuss our plan for presenting radiation topics in electromagnetic waves. Here is a broad outline in approximate, but not perfect order, as the insertion during lecture of an interesting problem out of some strict order or a math technique keeps things lively and helps to develop insight. Here is the outline.

	APPROACH	MAJOR TOPICS
MAXWELL'S EQUATION	DIRECT SOLN	PLANE WAVES MULTILAYERS
WAVE EQUATION	DIRECT SOLN	SEE PROPAGATION
FULL SPACE	VECTOR POTENTIAL(A, ϕ)	RADIATION IN FREE SPACE
	LORENZ GAUGE COULOMB GAUGE	SOURCE CURRENT SOURCE CHARGE ANTENNAS
	EXACT SOLUTIONS & RADIATION TERM	ELECTRIC DIPOLE MAGNETIC DIPOLE
HEMISPHERE	GREEN'S FUNCTION [RAYLEIGH, SOMMERFELD, SMYTHE]	EXTEND OPTICS 461 FOURIER OPTICS (OPT 564)
SCATTERING	CYLINDER: DIRECT SOLN $\hat{z}E_z$ SPHERE: DEBYE POTENTIALS	PLANE WAVE /WAVE EQUATION
WAVEGUIDE	METALLIC: HERTZ POTENTIALS FIBER:	

OTHER: SIGNAL REPRESENTATION, WHITE LIGHT, LAPLACE TRANSFORM

FOURIER TOPICS: RETARDED TIME $A(r;\nu)$, $\mathcal{A}(r;t)$

IDEAL XMSON FUNCTIONS: LENS

FREE SPACE PROPAGATION

INTRODUCTION TO VECTOR POTENTIAL - LAST LECTURE

STARTING WITH TIME DEPENDENT M.E.

$$\nabla \times \underline{E}(\underline{r},t) = -\frac{\partial \underline{B}(\underline{r},t)}{\partial t}$$

$$\nabla \times \underline{H}(\underline{r},t) = \underline{J} + \frac{\partial \underline{D}}{\partial t}$$

$\underline{A}(\underline{r},t)$

DEFINE VECTOR POTENTIAL $\underline{A}(\underline{r},t)$: $\underline{B} = \nabla \times \underline{A}(\underline{r},t)$

...

WE FOUND

$$\underline{E} = -\frac{\partial \underline{A}}{\partial t} - \nabla \phi$$

$$\boxed{\nabla^2 \underline{A}(\underline{r},t) - \mu\sigma \frac{\partial \underline{A}}{\partial t} - \mu\epsilon \frac{\partial^2 \underline{A}}{\partial t^2} = -\mu \underline{J}(\underline{r},t}$$

DO NOT BE PUT OFF BY THIS BRIEF PREVIEW. THE IDEA IS JUST TO ACQUAINT YOU WITH THE TYPES OF PDE WE WISH TO SOLVE. IT GIVES YOU A CHANCE TO THINK OVER - HOW WOULD YOU APPROACH THIS PROBLEM?

TEMPORAL $\mathcal{F}$ GIVES US

$$\boxed{\nabla^2 \underline{A}(\underline{r};\nu) - i\,2\pi\nu\mu\sigma\, \underline{A}(\underline{r};\nu) + (2\pi\nu)^2\mu\epsilon\, \underline{A}(\underline{r};\nu) = -\mu \underline{J}(\underline{r};\nu)}$$

VECTOR POTENTIAL $\underline{A}$, SCALAR Φ — TIME DEPENDENT FORM, LORENZ GAUGE

$$\nabla \times \underline{H} = \underline{J} + \epsilon \frac{\partial \underline{E}}{\partial t} \qquad \epsilon \nabla \cdot \underline{E} = \rho$$

$$\nabla \times \underline{E} = -\mu \frac{\partial \underline{H}}{\partial t} \qquad \nabla \cdot \underline{B} = 0$$

$$\underline{J} = \sigma \underline{E} \quad \text{ohm's Law}$$

SMYTHE $\sigma = \gamma$

$$-\nabla \times (\nabla \times \underline{H}) = \sigma\mu \frac{\partial \underline{H}}{\partial t} + \mu\epsilon \frac{\partial^2 \underline{H}}{\partial t^2} \quad \text{then}$$

$$-\nabla(\underbrace{\nabla \cdot \underline{H}}_{0}) + \nabla^2 \underline{H} = \sigma\mu \frac{\partial \underline{H}}{\partial t} + \mu\epsilon \frac{\partial^2 \underline{H}}{\partial t^2}$$

- $$\nabla^2 \underline{H} - \sigma\mu \frac{\partial \underline{H}}{\partial t} - \mu\epsilon \frac{\partial^2 \underline{H}}{\partial t^2} = 0$$

$$-\nabla \times (\nabla \times \underline{E}) = \mu \frac{\partial}{\partial t}\left[\underline{J} + \epsilon \frac{\partial \underline{E}}{\partial t}\right]$$

$$= \mu\sigma \frac{\partial \underline{E}}{\partial t} - \mu\epsilon \frac{\partial^2 \underline{E}}{\partial t^2}$$

- $$\nabla^2 \underline{E} - \sigma\mu \frac{\partial \underline{E}}{\partial t} - \mu\epsilon \frac{\partial^2 \underline{E}}{\partial t} = \nabla(\nabla \cdot E) = \nabla\left(\frac{\rho}{\epsilon}\right)$$

- LET $\underline{B} = \nabla \times \underline{A}$ — $\nabla \cdot A$ STILL ARBITRARY

$$\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t} = -\nabla \times \frac{\partial \underline{A}}{\partial t}$$

- $$\underline{E} = -\frac{\partial \underline{A}}{\partial t} - \nabla \phi \quad \Leftarrow \nabla \times \left(\underline{E} + \frac{\partial \underline{A}}{\partial t}\right) = 0$$

LORENZ GAUGE - DETAILS

SINCE $\underline{A}$: $\nabla \times A$ & $\nabla \cdot A$ SPECIFY

TAKE SM (7)417 11.01 :

- $$\nabla \cdot \underline{A} = -\mu\sigma\phi - \mu\epsilon \frac{\partial \phi}{\partial t}$$

$$\nabla(\nabla \cdot \underline{A}) = -\mu\sigma \nabla\phi - \mu\epsilon \frac{\partial \nabla\phi}{\partial t}$$

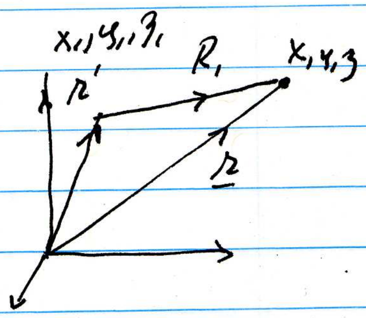
$$\nabla^2 \underline{A} + \nabla \times \nabla \times \underline{A} = +\mu\sigma\left(\underline{E} + \frac{\partial \underline{A}}{\partial t}\right) + \mu\epsilon \frac{\partial}{\partial t}\left(\underline{E} + \frac{\partial \underline{A}}{\partial t}\right)$$

$$\nabla^2 \underline{A} + \nabla \times \underline{B} = \nabla^2 \underline{A} + \cancel{\mu \underline{J}} + \cancel{\mu\epsilon \frac{\partial \underline{E}}{\partial t}}$$

$$= \cancel{\mu\sigma \underline{E}} + \cancel{\mu\epsilon \frac{\partial \underline{E}}{\partial t}} + \mu\sigma \frac{\partial \underline{A}}{\partial t} + \mu\epsilon \frac{\partial^2 \underline{A}}{\partial t^2}$$

- $$\nabla^2 \underline{A} = \mu\sigma \frac{\partial \underline{A}}{\partial t} + \mu\epsilon \frac{\partial^2 \underline{A}}{\partial t^2}$$ (Source free)

- $$\boxed{\nabla^2 \underline{A} - \mu\sigma \frac{\partial \underline{A}}{\partial t} - \mu\epsilon \frac{\partial^2 \underline{A}}{\partial t^2} = -\mu \underline{J}}$$
- $$\boxed{\nabla^2 \phi - \mu\sigma \frac{\partial \phi}{\partial t} - \mu\epsilon \frac{\partial^2 \phi}{\partial t^2} = -\frac{\rho}{\epsilon}}$$



SOURCES $\underline{J}(\underline{r}, t)$, $\rho(\underline{r}, t)$; $\sigma = 0$

$$\underline{A}(x,y,z,t) = \int_{\text{all space}} \frac{\underline{J}\left(x_1, y_1, z_1, t - \frac{R_1}{v}\right)}{R_1} \, dx_1 dy_1 dz_1 \qquad (1)$$

$$\phi(x,y,z,t) = \int_{\text{all space}} \frac{\rho\left(x_1, y_1, z_1, t - \frac{R_1}{v}\right)}{R_1} \, dx_1 dy_1 dz_1 \qquad (2)$$

Sm. 12.02

Vector Potential $\underline{A}$ and Scalar ϕ

Time Dependent Form

$$\nabla \times \underline{H} = \underline{J} + \epsilon \frac{\partial E}{\partial t} \qquad \epsilon \nabla \cdot \underline{E} = \rho$$

$$\nabla \times \underline{E} = -\mu \frac{\partial H}{\partial t} \qquad \nabla \cdot \underline{B} = 0$$

Vector Potential Lorenz Gauge $\quad \mu, \epsilon \quad \sigma = 0$

$$\underline{B} = \mu \underline{H} = \nabla \times \underline{A} \qquad \text{set } \nabla \cdot \underline{A} = -\mu\epsilon \frac{\partial \phi}{\partial t}$$

$$\nabla^2 \phi - \mu\epsilon \frac{\partial^2 \phi}{\partial t^2} = - \frac{\rho(r,t)}{\epsilon}$$

$$\nabla^2 \underline{A} - \mu\epsilon \frac{\partial^2 A}{\partial t^2} = -\mu \mathbb{J}(r,t)$$

SM 1.1202

$$\underline{A}(x,y,z,t) = \frac{\mu}{4\pi} \iiint \frac{J(x_1,y_1,z_1, t - \frac{R}{c})}{R} dx_1 dy_1 dz_1$$

$$\phi(x,y,z,t) = \frac{1}{4\pi\epsilon} \iint \frac{\rho(x_1,y_1,z_1, t - \frac{R}{c})}{R} dx_1 dy_1 dz_1$$

Using HTD or FT w r to time, the equations become

$$\underline{B}(r,\nu) = \mu \underline{H}(r,\nu) = \nabla \times A(r,\nu)$$

$$\nabla^2 \phi(r,\nu) + k^2 \phi(r,\nu) = -\frac{\rho(r,\nu)}{\epsilon}$$

$$\nabla^2 \underline{A}(r,\nu) + k^2 \underline{A} = -\mu \underline{J}(r,\nu)$$